La Red Temática CONACYT Agujeros Negros y Ondas Gravitatorias

… y algunos resultados específicos

XII Taller de la División de Gravitación y Física Matemática
Guadalajara, Jalisco

Noviembre 28, 2017

Olivier Sarbach

Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo

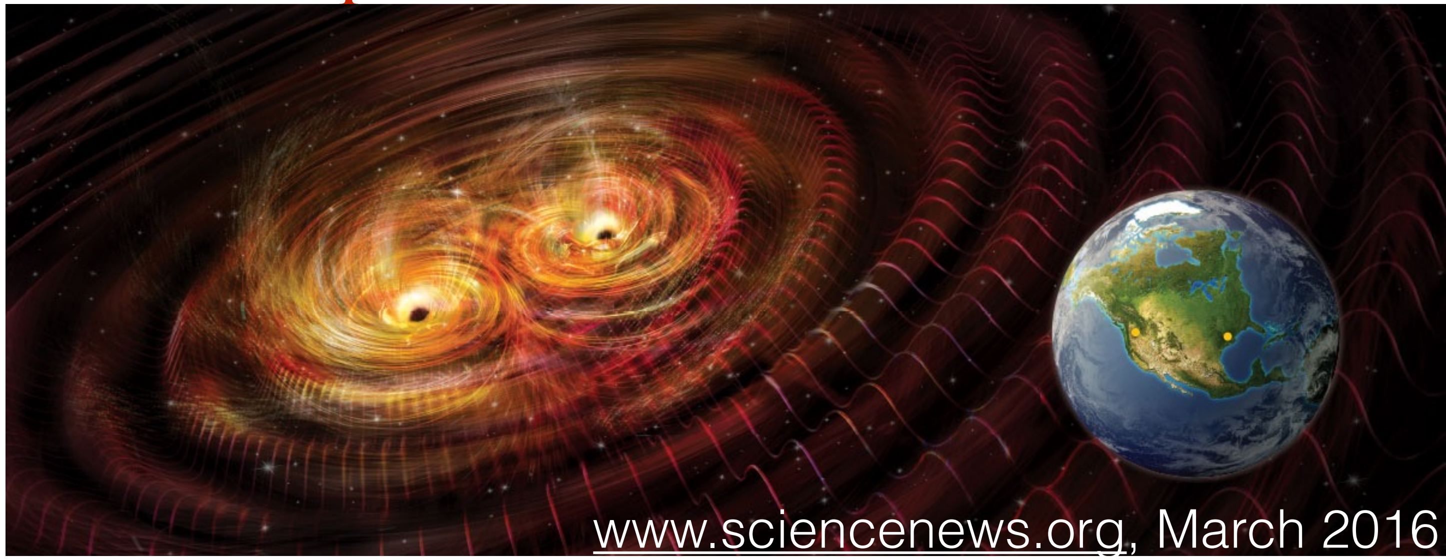
Morelia, México

Contenido

- Motivación: 100 años después de Einstein…
- Presentación de la Red Temática CONACYT
  “Agujeros Negros y Ondas Gravitatorias”
- Subtema: Acreción de un gas cinético relativista hacía un agujero negro
  Breve introducción a la teoría relativista de gases cinéticos
  Trayectorias no-acotadas y dispersión
  Trayectorias acotadas y el efecto “mixing”
- Conclusiones

100 años después de Einstein…

11 de febrero 2016: Las colaboraciones LIGO y VIRGO anuncian la primera detección directa de ondas gravitacionales. La señal proviene de un sistema binario de agujeros negros que fusionaron hace ~1300Myr.
Desde entonces se han medido otras 4 señales provenientes de sistemas binarios de agujeros negros.



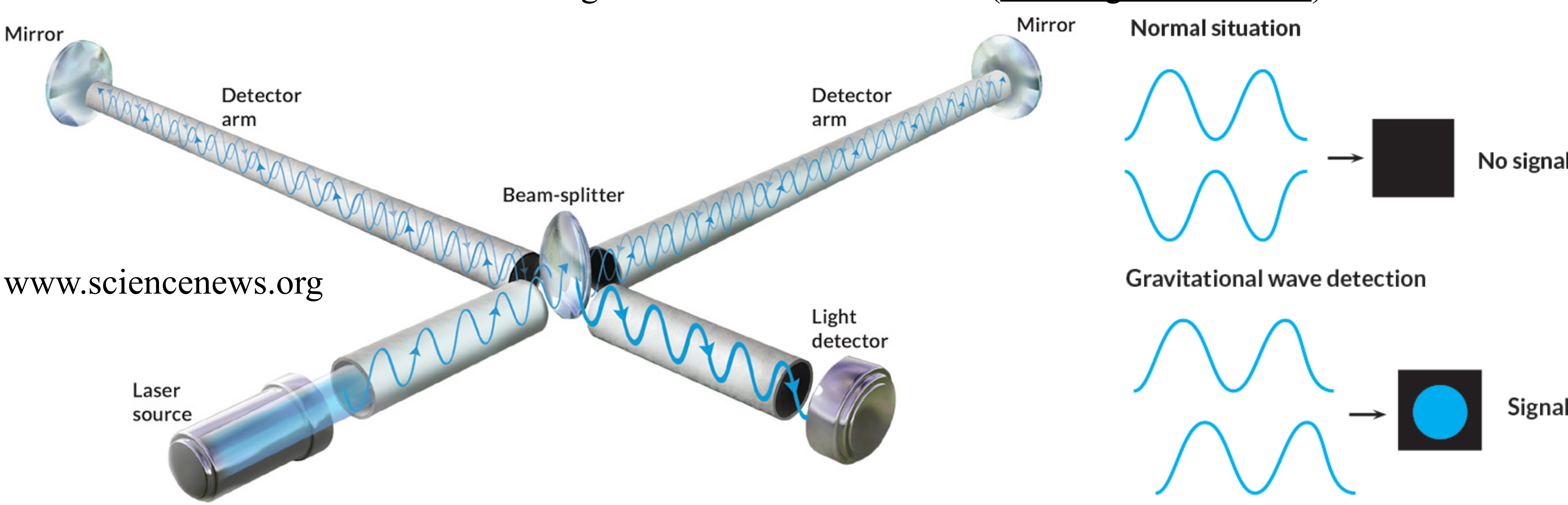
www.sciencenews.org, March 2016

3 de octubre 2017: Premio Nobel de Física a Rainer Weiss, Barry Barish y Kip Thorne "for decisive contributions to the LIGO detector and the observation of gravitational waves".

¡Una nueva ventana sobre nuestro universo!

100 años después de Einstein…

LIGO Livingston and VIRGO detectors (www.ligo.caltech.edu)

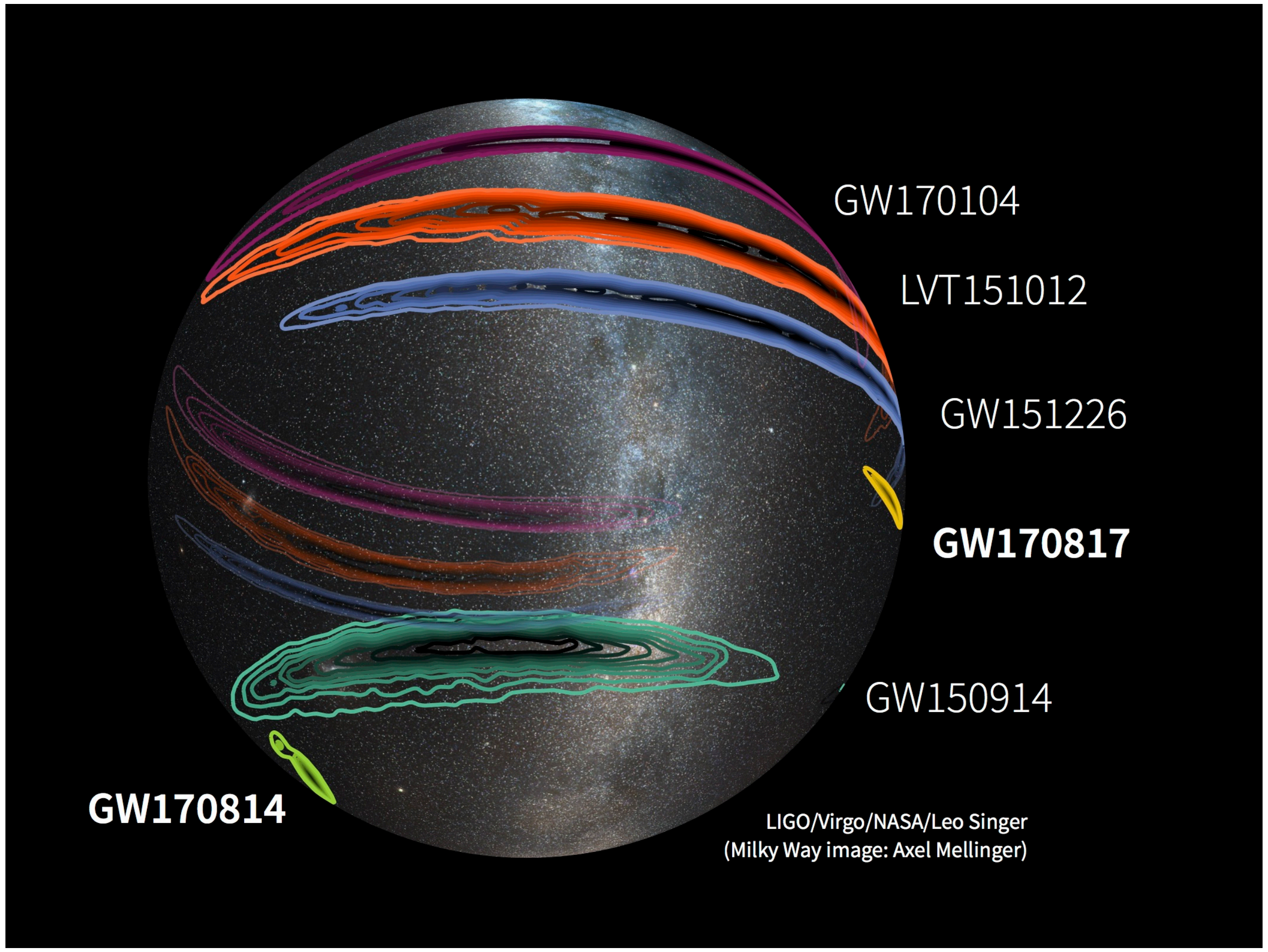


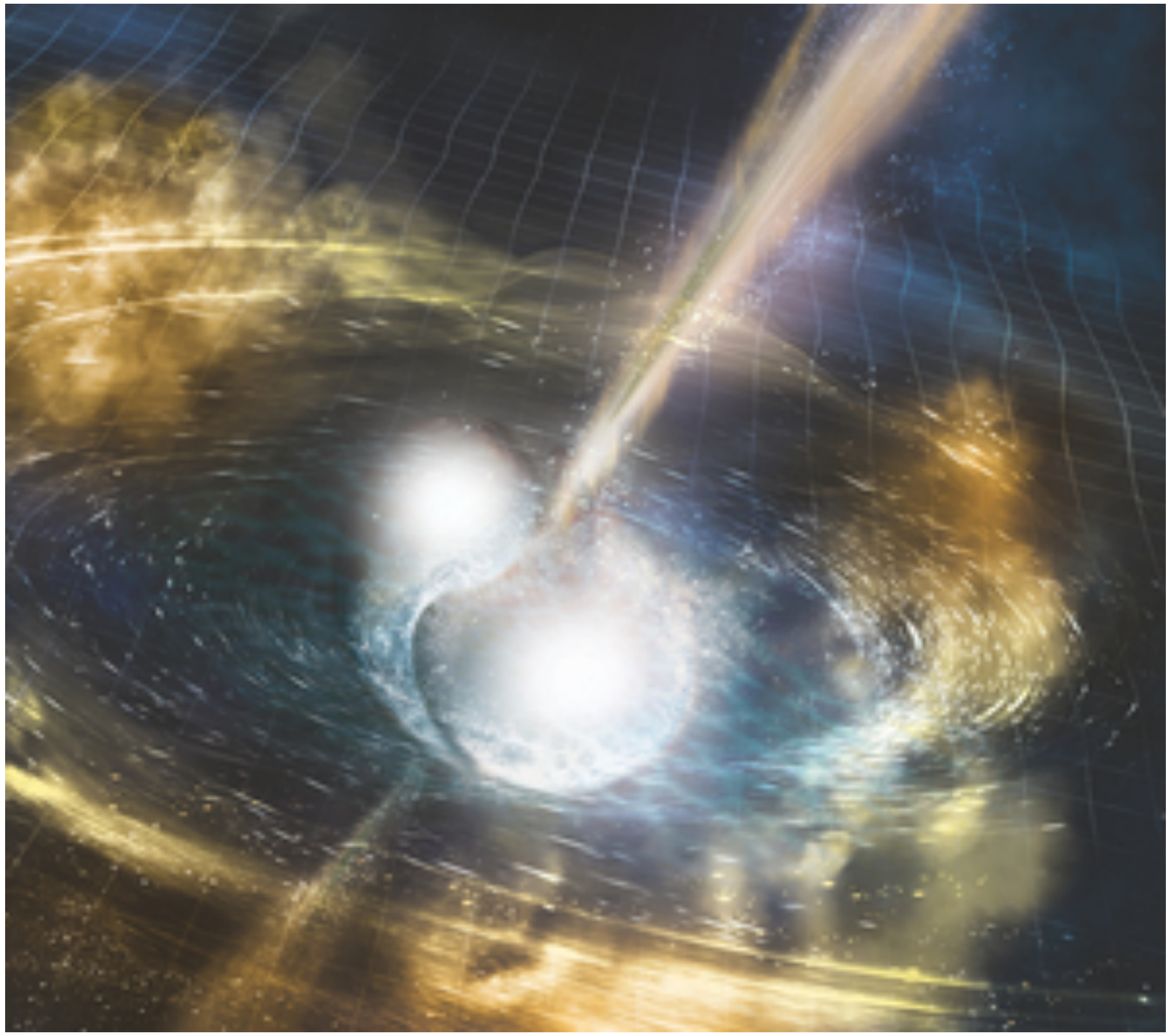
www.sciencenews.org

Deformación inducida = medir la distancia entre Mérida y Tijuana con la precisión del radio del proton!

100 años después de Einstein…

16 de octubre 2017: Primera detección de un sistema binario de estrellas de neutrones. Se detectó no solamente en el espectro de ondas gravitacionales, sino también en el espectro electromagnético (rayos gamma, rayos X, óptico, etc.). "Multi-messenger astronomy"





www.ligo.caltech.edu

Historia de la Red (2016)

- Pocos días después del 11 de febrero de 2016: Claudia Moreno propone (a Darío Núñez y a mi) la formación de una red temática CONACYT "¡vámonos, es fácil!".

- *Febrero 22, 2016*: Se envía el proyecto (pocas horas antes de la fecha límite):
  Red Temática de Agujeros Negros Vibrantes y Emisión de Ondas Gravitatorias
  Aprobado por un monto de $951,000 para 2016.

- 19 miembros (la red temática CONACYT más pequeña de 2016!)
  Miguel Alcubierre, Javier Antelis, Eliana Chaverra, Juan Carlos Degollado, Max Dohse, Ramiro Franco, Jaime Mendoza, Manuel Altamirano, Claudia Moreno, Darío Núñez, Néstor Ortiz, Malik Rakhmanov, Paola Rioseco, Carlos Rodríguez, Roberto Santos, Olivier Sarbach, Manuel Tiglio, Guillermo Valdés, Thomas Zannias.

- Temas de investigación prioritarios:
  Análisis de datos de interferometría láser
  Modos cuasi-normales de agujeros negros
  Análisis matemático del problema de valores iniciales y de contorno de las ecs. de Einstein
  Acreción de materia hacía agujeros negros
  Modelos de halos de materia oscura alrededor de agujeros negros

Red ANyOG

Historia de la Red (2017)

- *Enero 20, 2017*: Se envió el proyecto (pocas horas antes de la fecha límite):

  Red Temática de Agujeros Negros ~~Vibrantes~~ y ~~Emisión de~~ Ondas Gravitatorias

  Aprobado con un monto de ~~\$951,000 (2016)~~ \$870,000 para el 2017

  (poner en contexto: 23% de reducción del presupuesto CONACYT para el 2017)

- 67 miembros (la red temática CONACYT ~~más pequeña~~)
  (más o menos 50% de estudiantes)

- Temas de investigación prioritarios:
  Análisis de datos de interferometría láser
  Modos cuasi-normales de agujeros negros
  Análisis matemático del problema de valores iniciales y de contorno de las ecs. de Einstein
  Acreción de materia hacía agujeros negros
  Modelos de halos de materia oscura alrededor de agujeros negros
  Agujeros negros primordiales
  Aspectos teóricos de los agujeros negros

Red ANyOG

Logros (2016)

- Se hizo una página web www.redtematicaanyog.mx con información sobre la red, una lista de miembros, eventos, publicaciones, software, etc. *(in Spanish, English version soon!)*
- Taller de campos escalares y halos de materia oscura alrededor de agujeros negros súpermasivos (Morelia, Junio 3-5, 2016)
- Financiamiento para varios miembros (estudiantes and profesores) para asistir a la 21st International Conference on General Relativity and Gravitation en Nueva York, Julio 2016
(1 charla paralela y 3 carteles presentados por miembros de la red)
- Financiamiento para asistir a la Segunda Escuela Centroamericana de Física Fundamental, Tegucigalpa, agosto 26-septiembre 2, 2016
(curso introductorio sobre la física de los agujeros negros presentado por un miembro)
- Reunión Anual de Redes CONACYT, Cuernavaca, octobre 5-7, 2016
- Congreso Nacional de Física en León, Guanajuato, octubre 3-7, 2016
(estudiante miembro Roberto Esparza ganó un premio para la presentación de su cartel!)
- Primera Escuela de Relatividad General y Ondas Gravitatorias (ERGOG I) y Primera Reunión de la Red, Guadalajara, noviembre 2016 (> 80 estudiantes y > 20 profesores!)
- 6 artículos publicados con dos o más co-autores miembros del proyecto,
3 carteles, un artículo de divulgación y 1 tesis de doctorado, 1 de licenciatura

Red ANyOG

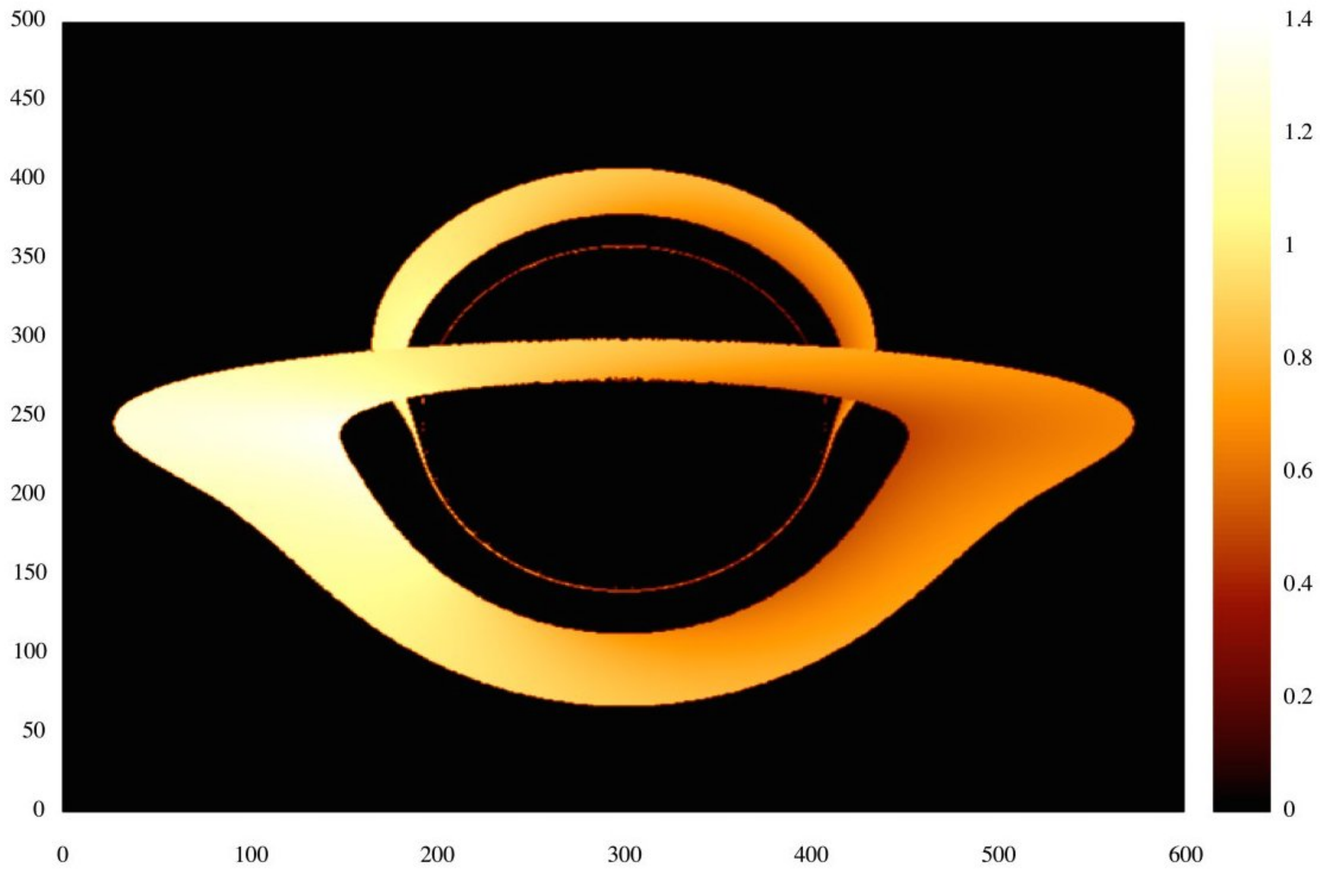
Logros (2017, hasta el día)

- Se aumentó el número de miembros de 19 a 68
- Taller de campos escalares y halos de materia oscura alrededor de agujeros negros súpermasivos (Guanajuato, Junio 15-17, 2017)
- Financiamiento para varios miembros (estudiantes y profesores) para su asistencia a:
  North American Einstein Toolkit School and Workshop at NCSA, Urbana, EEUU, agosto 3-4, 2017
  Congreso Nacional de Física, Monterrey, octubre 9-13, 2017
  (estudiante miembro Kenia Ramírez ganó un premio para la presentación de su cartel!)
  Early Results from GW Searches and Electromagnetic Counterparts, IAU Symposium, Baton Rouge, EEUU, octubre 16-19, 2017
  XII Taller de la División de Gravitación y Física Matemática, Guadalajara, noviembre 27-diciembre 1, 2017
- Segunda Escuela de Relatividad General y Ondas Gravitatorias (ERGOG II) en Cuernavaca, agosto 7-9, 2017 (~50 estudiantes)
- Reunión Anual de Redes CONACYT, Xochiltepec, agosto 31-septiembre 1, 2017
- Segunda Reunión de la Red en Guadalajara, noviembre 6-7, 2017 y curso especial *Data Analysis and Signal Processing*, Guadalajara, noviembre 8-10, 2017
  (33 estudiantes y 11 profesores, 40% mujeres)

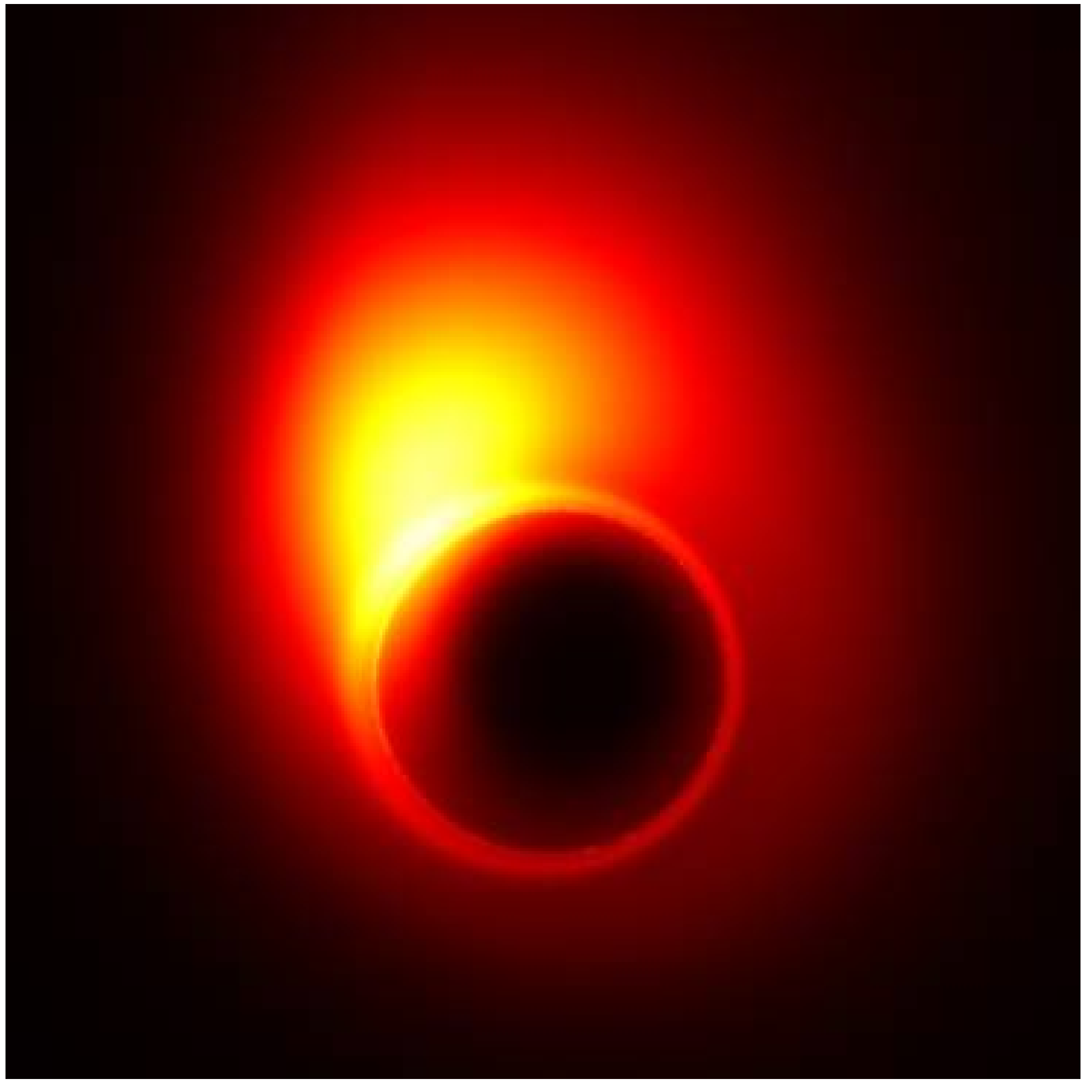
Red ANyOG

Subtema: Acreción de un gas cinético relativista hacía un agujero negro

Event Horizon Telescope: Observing the Black Hole Shadow



Calculated image of stationary disc surrounding a black hole: Strong lensing and black hole shadow
(Enrique Vizcarra, 2015)



Model for the supermassive black hole at the center of M87

(A.E. Broderick,
Event Horizon Telescope)

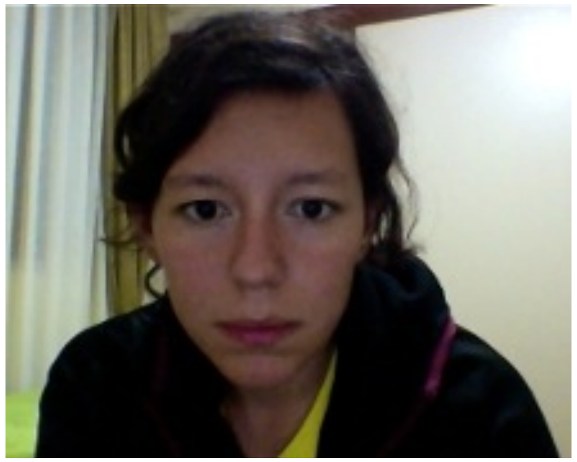
Relativistic kinetic theory

- Provides a more fundamental description for a (dilute) gas/dust fluid
- Has important applications in astrophysics (accretion of dilute gas into a black hole, distribution of stars around supermassive black holes) and cosmology (early universe)
- Phase space concept becomes important
  (brings together general relativity and statistical mechanics!)
- Leads to the *general relativistic* Boltzmann equation
  (Israel, Lindquist, Stewart, Ehlers, Cercignani & Kremer, …)
- Mathematical questions: well-posedness, global existence, static and stationary solutions (Rein & Rendall, Andréasson, Kunze, Dafermos, Ringström, Fajman, Joudioux, Smulevici, Taylor, …)

General covariance requires *geometric formulation* of the theory!

Relativistic kinetic theory: Motivation

Our motivation was to provide a fully covariant (on phase space) formulation of the theory by exploiting the rich geometric structures on the (co)tangent bundle associated with the spacetime manifold.



Paola Rioseco (IFM)

Thomas Zannias (IFM)

Based on joint work with Paola Rioseco and Thomas Zannias
AIP Conf. Proc. 1548 (2013) 134-155
Class. Quantum Grav. 31, 085013 (2014)
Class. Quantum Grav. 34, 095007 (2017)
J. Phys. Conf. Ser. 831 (2017)

Geometry of the cotangent bundle

Geometric formulation for a simple gas of identical particles of mass m > 0

- Configuration space: spacetime manifold (M, g) (time-oriented)

- Relativistic phase space:
  Cotangent bundle: $T^*M = \{(x,p) : x \in M, p \in T^*_x M\}$
  Mass shell: $\Gamma_m = \{(x,p) \in T^*M : g_x^{-1}(p,p) = -m^2, p \text{ future-directed}\}$

- Symplectic two-form: $\Omega_{(x,p)} = dp \circ \pi_{*(x,p)} = dp_\mu \wedge dx^\mu$

- Free particle Hamiltonian: $H(x,p) := \frac{1}{2} g_x^{-1}(p,p)$

- Liouville vector field L is defined as corresponding Hamiltonian vector field $dH = \Omega(\cdot, L)$:

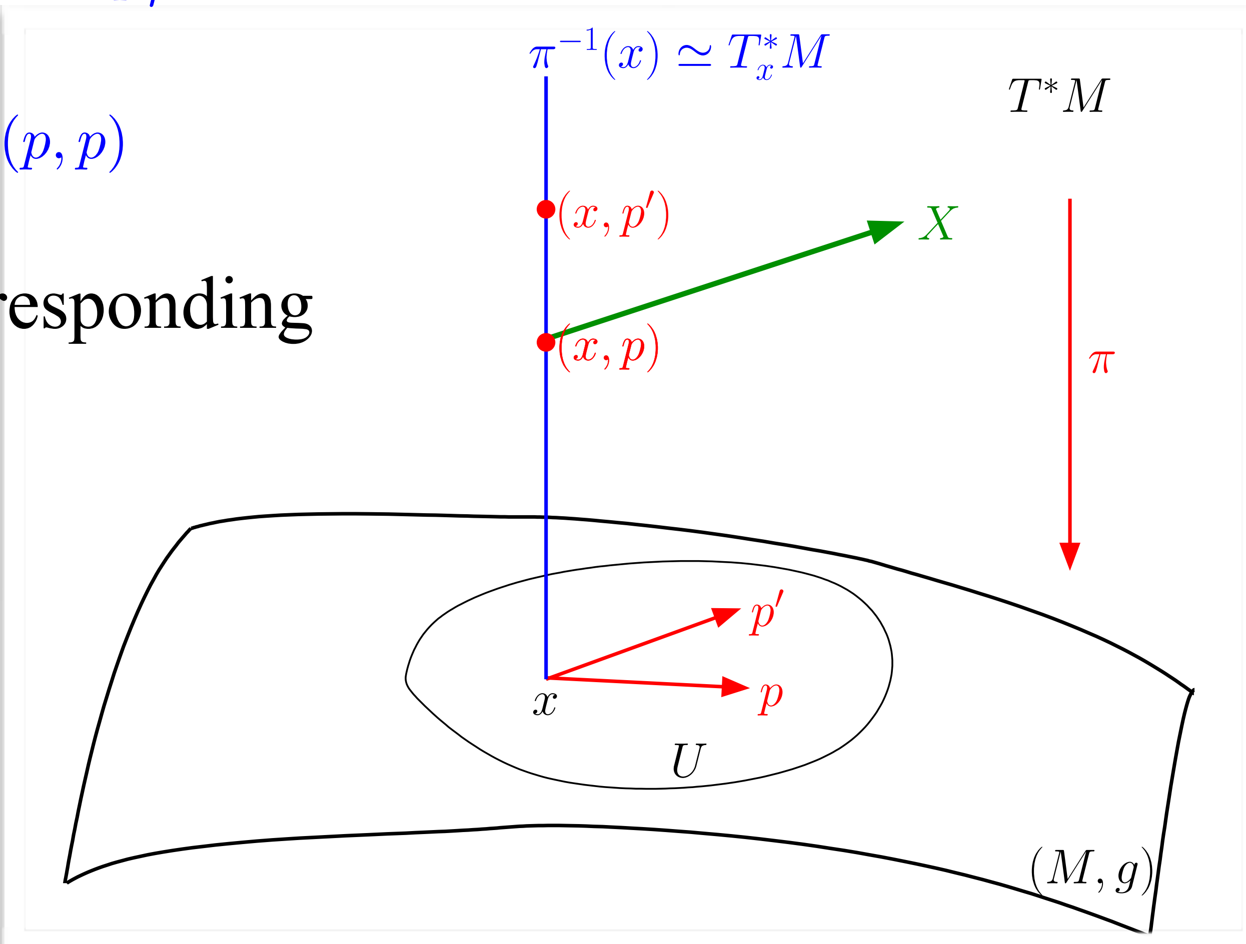
$$L = g^{\mu\nu}(x) p_\nu \frac{\partial}{\partial x^\mu} - \frac{1}{2} p_\alpha p_\beta \frac{\partial g^{\alpha\beta}(x)}{\partial x^\mu} \frac{\partial}{\partial p_\mu}$$

- Relativistic Boltzmann equation:

$$L[f] = C(f,f)$$

collision term

one-particle distribution function



Liouville vector field

$$L = g^{\mu\nu}(x)p_\nu \frac{\partial}{\partial x^\mu} - \frac{1}{2}p_\alpha p_\beta \frac{\partial g^{\alpha\beta}(x)}{\partial x^\mu}\frac{\partial}{\partial p_\mu}$$

Integral curves of L

$$\dot{x}^\mu = g^{\mu\nu}p_\nu, \quad \dot{p}_\mu = -\frac{1}{2}p_\alpha p_\beta \frac{\partial g^{\alpha\beta}}{\partial x^\mu}$$

When projected on (M,g) they describe spacetime geodesics.

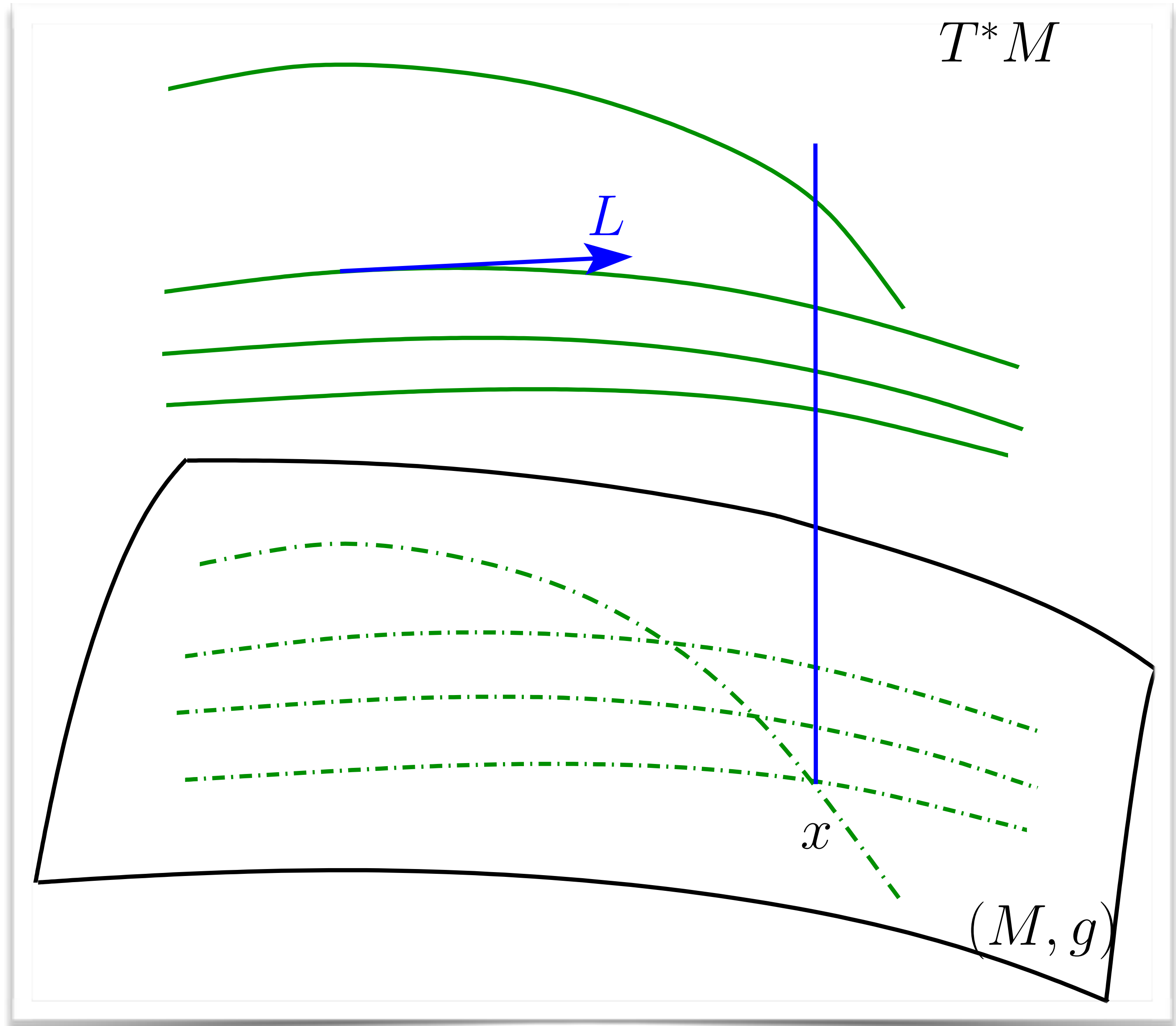
Liouville's theorem

- Volume form $\eta := \Omega \wedge \Omega \wedge \Omega \wedge \Omega$
- By definition $i_L\Omega = -dH$ hence

  $$\pounds_L \Omega = di_L\Omega + i_L d\Omega = -d^2 H = 0$$
- This immediately implies

  $$\pounds_L \eta = 0$$

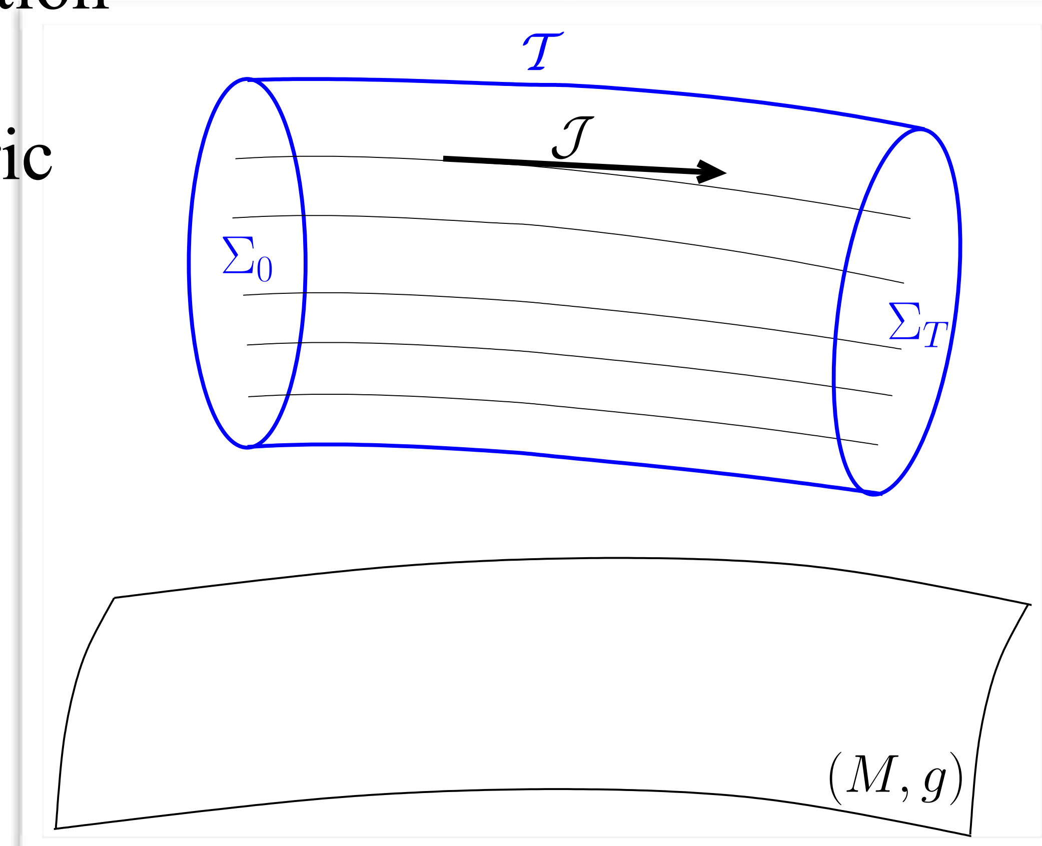
  and hence L is divergence-free.



Liouville vector field has no critical points when restricted to the mass shells with m > 0!

Relativistic kinetic theory

Physical interpretation of the distribution function

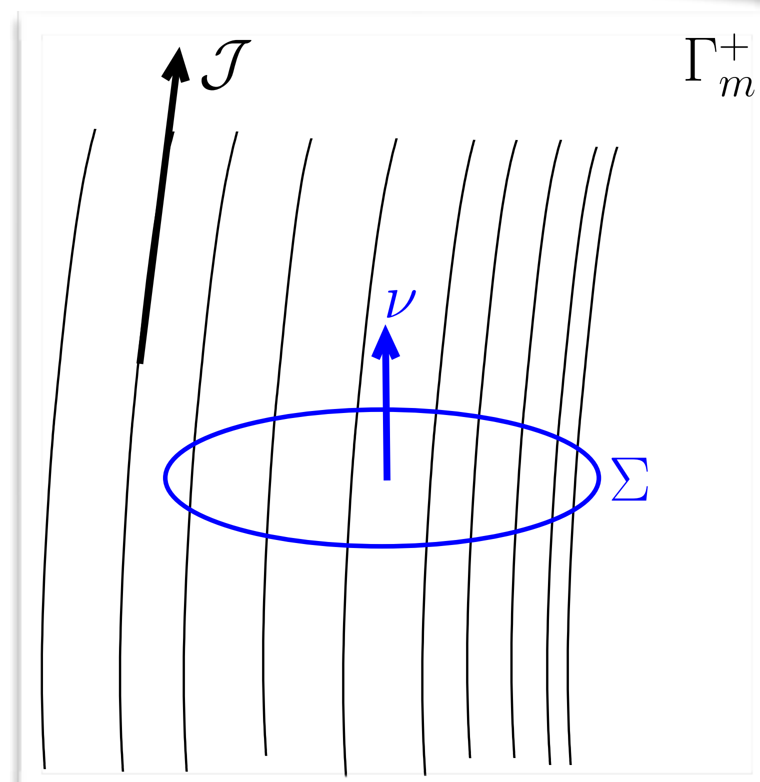


- Spacetime metric g induces naturally a metric G on the cotangent bundle (Sasaki metric)
- 7-velocity field: $U := \frac{1}{m} L$ on mass shell Γ_m divergence-free due to Liouville's theorem
- 7-current density vector field:

  $\mathcal{J} := fU$ on mass shell Γ_m
- Let Σ be a 6-dimensional spacelike hypersurface with normal vector ν. Then, the flux integral

  $$N[\Sigma] = -\int_\Sigma G(\mathcal{J}, \nu) d\Sigma$$

  is the averaged number of occupied trajectories that intersect Σ.



OS & T. Zannias, Class. Quantum Grav. 31, 085013 (2014)

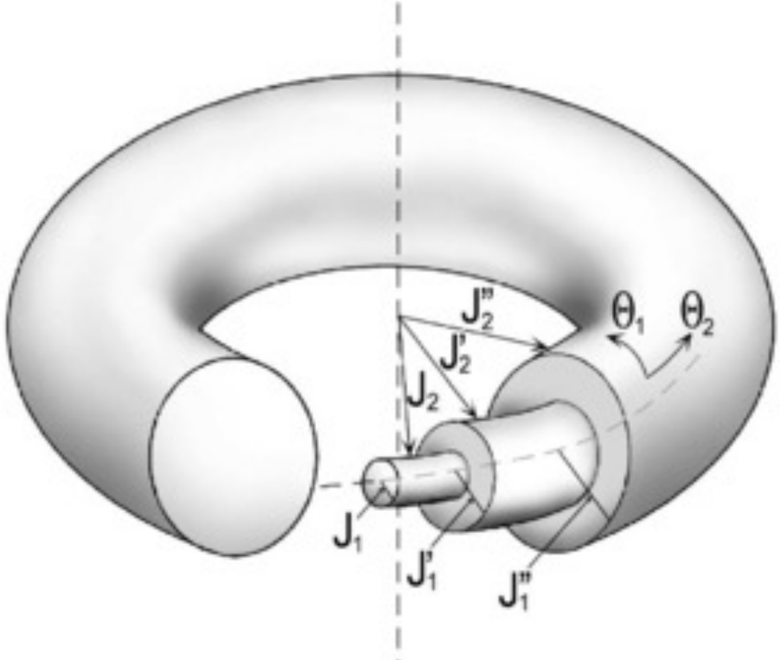
Explicitly solving the Liouville equation on a black hole background

- Hamiltonian system H on 2n-dimensional phase space is called integrable if there exists n conserved quantities $F_1 = H, F_2, F_3, \ldots, F_n$ such that

$$\{F_i, F_j\} = 0, \qquad dF_i \text{ pointwise linearly independent}$$

- Invariant submanifolds $\Gamma(c_1, \ldots, c_n) := \{(x,p) \in T^*M : F_i(x,p) = c_i, i = 1, \ldots, n\}$

- Usually (i.e. if compact) $\Gamma(c_1, \ldots, c_n) = T^n$ a torus, action-angle variables (Q,J) can be defined and motion characterized by frequencies on the torus.



cdn.iopscience.com

- Free-particle Hamiltonian on Schwarzschild is integrable ($F_2 = E, F_3 = L_z, F_4 = L^2$)

- Free-particle Hamiltonian on Kerr is also integrable! ($F_2 = E, F_3 = L_z, F_4 =$ Carter constant)

- Invariant submanifold $\mathbb{R} \times S^1 \times S^1 \times S^1$ or $\mathbb{R}^2 \times S^1 \times S^1$ depending on whether motion bounded or unbounded. Not compact!

Explicitly solving the Liouville equation on a black hole background

- The Q variables in the unbounded case

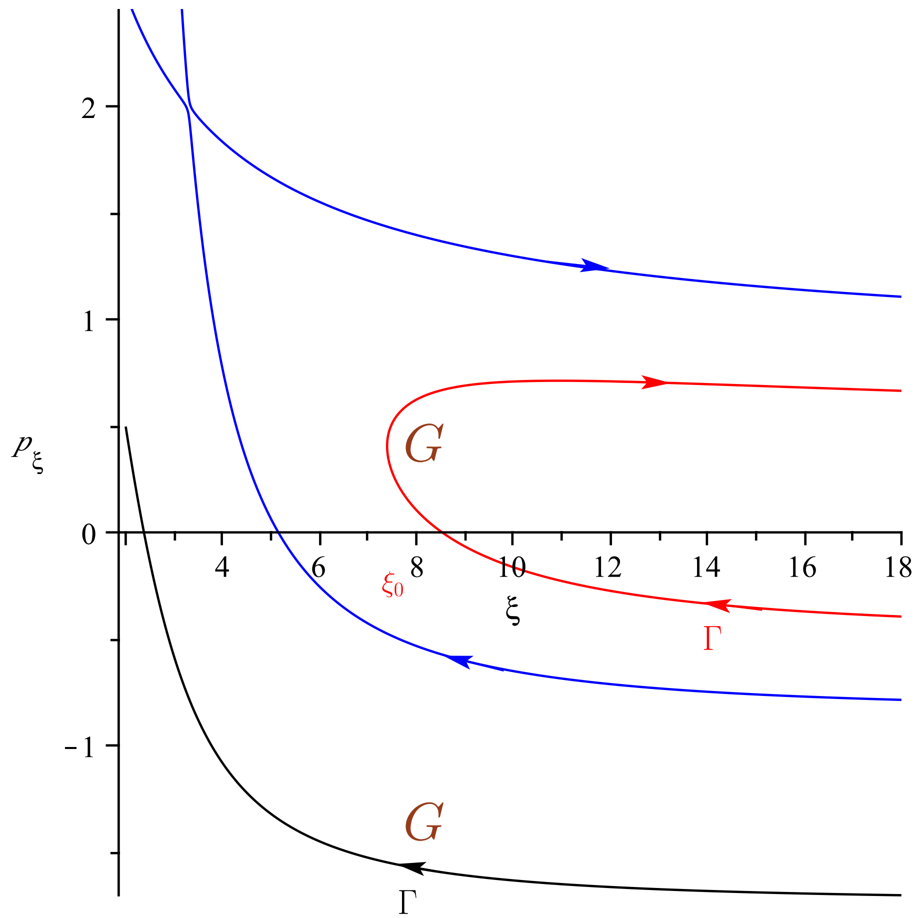
$$Q^0 := \frac{\partial S}{\partial m} = -m \int\limits_{\gamma_{(r,p_r)}} \frac{dr}{\left(1-\frac{2}{r}\right)p_r - \frac{2}{r}E},$$

G

$$Q^1 := \frac{\partial S}{\partial E} = -t + \int\limits_{\gamma_{(r,p_r)}} \frac{\frac{2}{r}p_r + \left(1+\frac{2}{r}\right)E}{\left(1-\frac{2}{r}\right)p_r - \frac{2}{r}E}dr,$$

$$Q^2 := \frac{\partial S}{\partial \ell_z} = \varphi - \ell_z \int\limits_{\gamma_{(\vartheta,p_\vartheta)}} \frac{d\vartheta}{p_\vartheta \sin^2\vartheta},$$

$$Q^3 := \frac{\partial S}{\partial \ell} = -\ell \int\limits_{\gamma_{(r,p_r)}} \frac{dr}{r^2\left[\left(1-\frac{2}{r}\right)p_r - \frac{2}{r}E\right]} + \ell \int\limits_{\gamma_{(\vartheta,p_\vartheta)}} \frac{d\vartheta}{p_\vartheta}.$$



Q^2 and Q^3 are 2π-periodic along the curve $\gamma_{(\vartheta,p_\vartheta)}$ (angle-variables).

Explicitly solving the Liouville equation on a black hole background

- In terms of the new symplectic (action-angle-like) variables (Q,P) the Liouville vector field is trivialized, so

$$L[f] = \frac{\partial}{\partial Q^0}[f] = 0$$

- Most general solution describing the dynamics of a collisionless gas propagating on the curved geometry of a black hole background:

$$f(x,p) = F(Q^1, Q^2, Q^3, P_0, P_1, P_2, P_3)$$

t + ... $\varphi + \dots$

- Stationary and axisymmetric if and only if independent of the first two variables

- Spacetime observables (current density and stress-energy momentum tensor):

$$J_\mu(x) = \int\limits_{\pi^{-1}(x)} p_\mu f(x,p) \frac{d^4p}{\sqrt{-g}}, \qquad T_{\mu\nu}(x) = \int\limits_{\pi^{-1}(x)} p_\mu p_\nu f(x,p) \frac{d^4p}{\sqrt{-g}}$$

$$T_{\mu\nu} = T_{\mu\nu}^{(abs)} + T_{\mu\nu}^{(scat)} + T_{\mu\nu}^{(bounded)}$$

bounded trajectories

absorbed by BH

scattered particles

Spherical steady-state solutions (unbounded trajectories)

- For a spherical, steady-state solution

$$f(x,p) = F(m,E,\ell)$$

- If the gas is accreted from a reservoir of identical particles at infinity which is in equilibrium, then at infinity (and hence everywhere)

$$f(x,p) = \alpha\delta\left(\sqrt{-2H(x,p)} - m\right)e^{-\beta E}, \quad \beta\text{: inverse temperature}$$

- Dimensionless quantity (typically very large):

$$z := m\beta = \frac{mc^2}{k_B T} \gg 1$$

- Decomposition of the observables:

$$J^\mu = n u^\mu, \qquad T^{\mu\nu} = \varepsilon e_0^\mu e_0^\nu + \sum_{j=1,2,3} p_j e_j^\mu e_j^\nu$$

particle density **energy density** **principle pressures**

In general, e_0^μ is not proportional to u^μ, and $p_1 \neq p_2 \neq p_3$.

Spherical steady-state solutions (unbounded trajectories)

- Observables have three contributions:

$$T_{\mu\nu} = T_{\mu\nu}^{(abs)} + T_{\mu\nu}^{(scat)} + T_{\mu\nu}^{(bounded)}$$

absorbed by BH
no contribution at infinity

scattered particles
no contribution at horizon

bounded trajectories
no contribution at horizon nor infinity

- At infinity, gas behaves like isotropic fluid with ideal gas equation of state

$$n_\infty(z) = 4\pi\alpha m^4 \frac{K_2(z)}{z}, \qquad \varepsilon_\infty(z) = 4\pi\alpha m^5 \left[\frac{K_1(z)}{z} + \frac{3K_2(z)}{z^2}\right], \qquad p_\infty(z) = \beta^{-1} n_\infty(z)$$

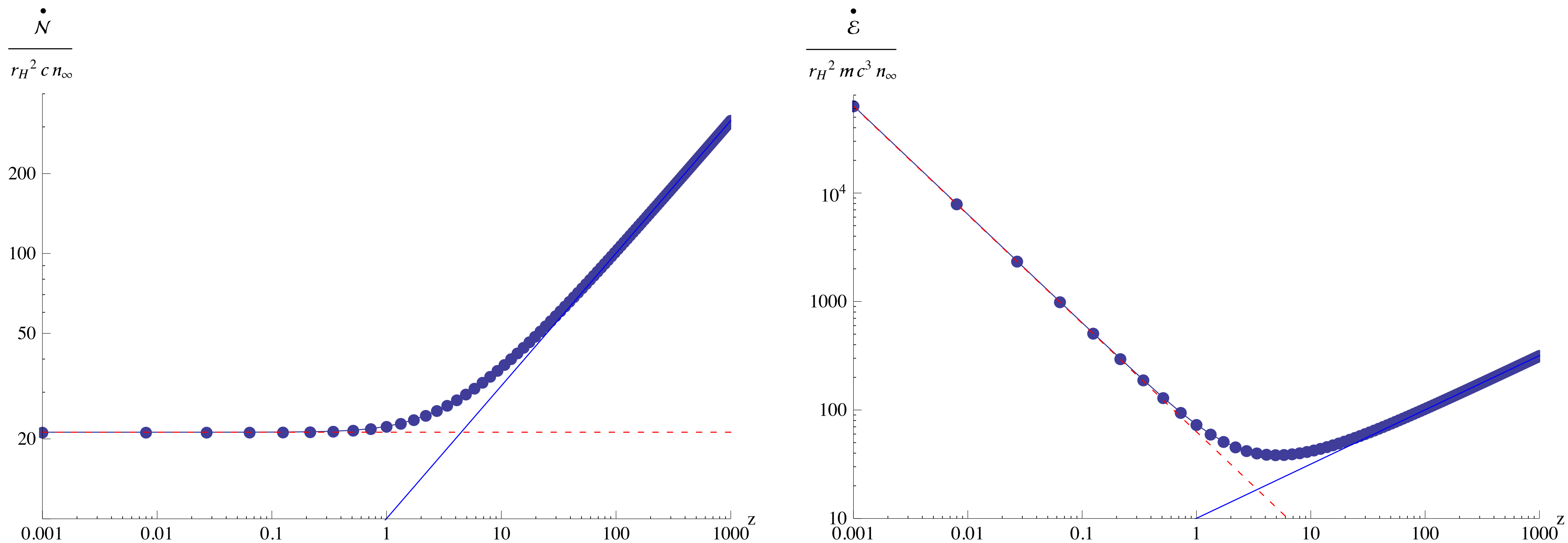
- At the horizon, the gas ceases to be isotropic. For large values of z:

$$\lim_{z\to\infty} \frac{\varepsilon_H}{n_H} = \frac{mc^2}{2\sqrt{3}}\left(3 + \sqrt{\frac{31}{3}}\right) \simeq 1.79mc^2$$

$$\lim_{z\to\infty} \frac{p_{rad}}{n_H} = \frac{mc^2}{2\sqrt{3}}\left(-3 + \sqrt{\frac{31}{3}}\right) \simeq 0.0619mc^2, \quad \lim_{z\to\infty} \frac{p_{tan}}{n_H} = \frac{mc^2}{\sqrt{3}} \simeq 0.677mc^2$$

The tangential pressure is about nine times as large as the radial one!

Accretion rate (spherical steady-states)



Comparison between particle and energy accretion rates as a function of inverse temp. z

- For low temperatures (agrees with Newtonian calculations by Shapiro & Teukolsky)

$$\dot{\mathcal{E}} \sim 4 r_H^2 m c^3 \sqrt{2\pi z}\, n_\infty \simeq (2.2 \times 10^{-21}) \left(\frac{M_H}{10 M_\odot}\right)^2 \left(\frac{m}{m_p}\right) \left(\frac{n_\infty}{1cm^3}\right) \left(\frac{z}{10^9}\right)^{1/2} \frac{M_\odot c^2}{yr},$$

using typical values for the ionized component of the interstellar medium in our Galaxy. (smaller by a factor z compared to hydrodynamic case of Michel accretion!)

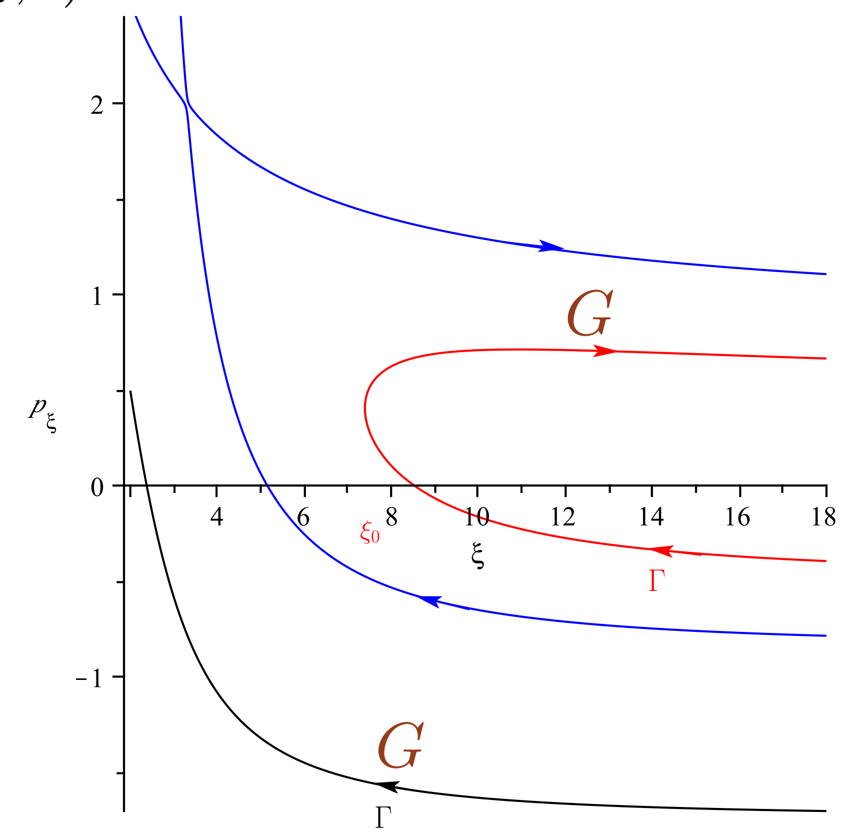
- For high temperatures particle rate goes to a constant but energy rate diverges like 1/z.

Stability theorem (unbounded trajectories)

- How does the flow behave if we keep reservoir at infinity, but perturb the distribution function inside a compact region within initial hypersurface?

- Suppose initial distribution function *F* satisfies the following conditions:
  (1) $F = 0$ for bounded orbits
  (2) $0 \leq F \leq \alpha e^{-\beta E}$ (i.e. bounded by an equilibrium distribution function)
  (3) $\lim\limits_{G \to \pm\infty} F(G, Q^2, Q^3, E, \ell_z, \ell) = f_\infty(E)$ uniformly in (Q^2, Q^3, ℓ_z, ℓ)

  Then, along the world lines of static observers, the components of the current density and stress energy-momentum with respect to a non-rotating frame along the observer converge pointwise in time to corresponding observables of the steady-state solution with distribution function $f_\infty(E)$.



- In particular, if $f_\infty(E) = 0$ the gas disperses completely.

- Proof is simple consequence of Lebesgue's dominated convergence theorem.

Bounded trajectories: Ongoing work

- Currently analyzing time asymptotics of solutions associated with bounded trajectories in a Kerr spacetime.

- Motion is quasi-periodic, so just boring?

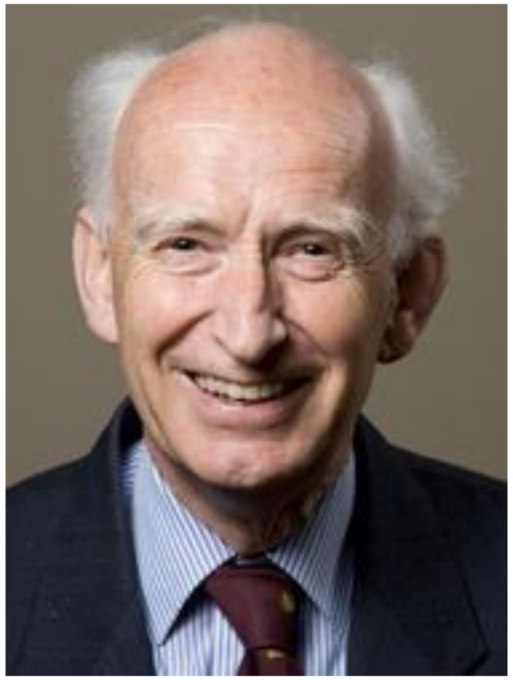
- Far from it! Even though the motion is quasi-periodic, there seems to be a relaxation phenomena taking place due to phase-space mixing!

Bounded trajectories: Ongoing work

- Currently analyzing time asymptotics of solutions associated with bounded trajectories in a Kerr spacetime.

- Motion is quasi-periodic, so just boring?

- Far from it! Even though the motion is quasi-periodic, there seems to be a relaxation phenomena taking place due to phase-space mixing!
  D. Lynden-Bell 1962: galactic dynamics
  C. Villani & C. Mouhot 2011: Landau damping



www.kavliprize.org

www.francetvinfo.fr

Phase space mixing: toy model example

- Consider first the harmonic oscillator in 1D:

Hamiltonian function and symplectic form:

$$H(x,p) = \frac{1}{2}(x^2 + p^2), \qquad \Omega = dp \wedge dx$$

- In terms of action-angle variables $(x,p) = \sqrt{2E}(-\cos Q, \sin Q)$

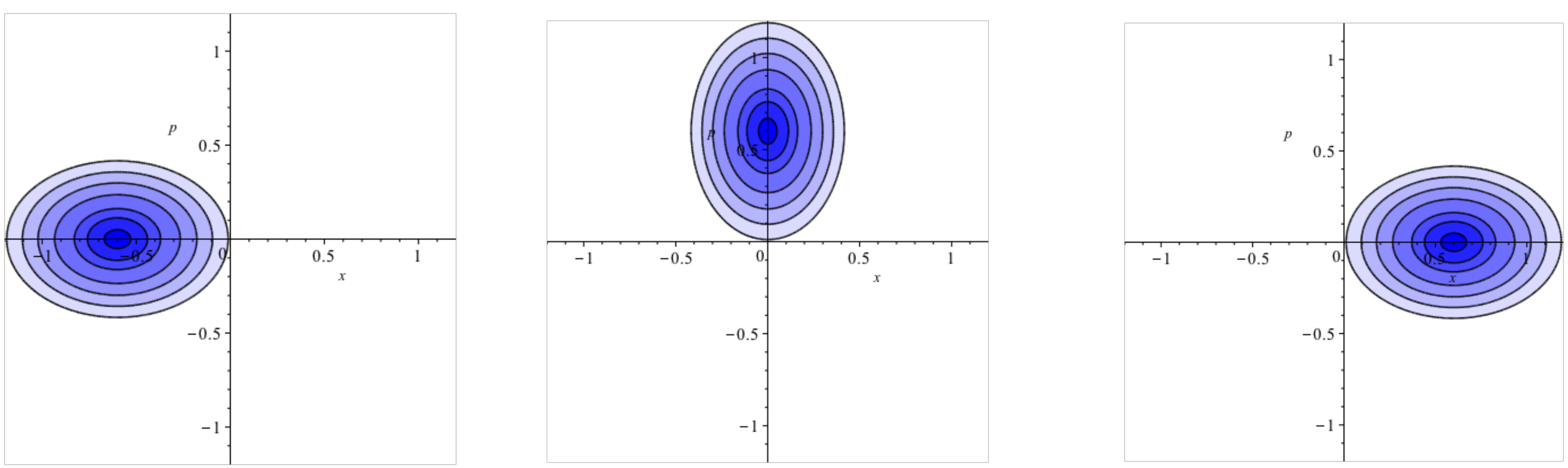
Hamiltonian function and symplectic form:

$$H(Q,E) = E, \qquad \Omega = dE \wedge dQ$$

Equations of motion:

$$\dot{Q} = \frac{\partial H}{\partial E} = 1 = \omega, \qquad \dot{E} = -\frac{\partial H}{\partial Q} = 0$$

- Initial distribution of states just rotates in phase along the energy curves:



Phase space mixing: toy model example

- Change the period of each energy surface slightly:

Hamiltonian function and symplectic form:

$$H(x,p) = F\left[\frac{1}{2}(x^2+p^2)\right], \quad F(E) = E + \frac{k}{2}E^2, \qquad \Omega = dp \wedge dx$$

- In terms of action-angle variables $(x,p) = \sqrt{2E}(-\cos Q, \sin Q)$

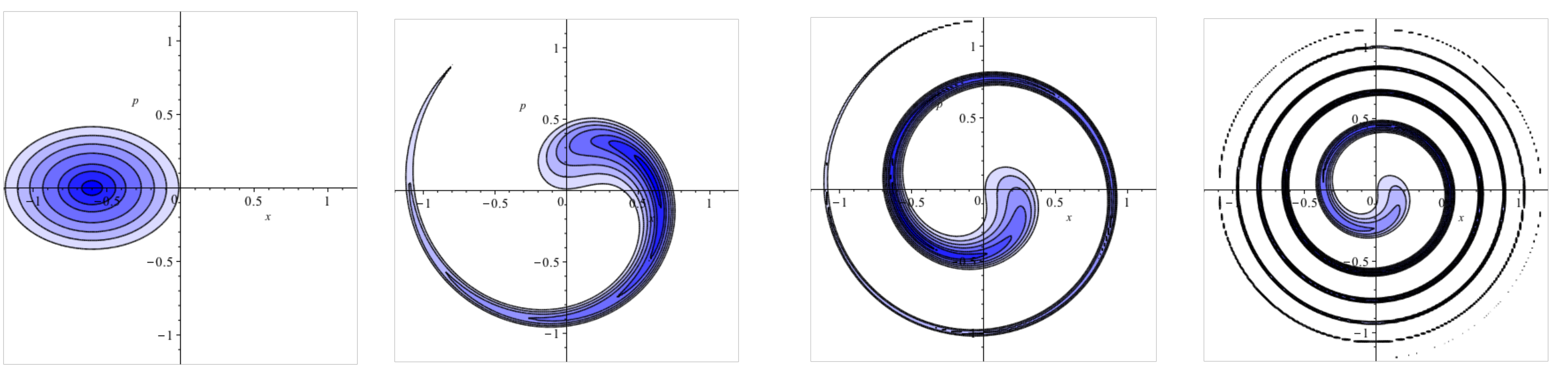
Hamiltonian function and symplectic form:

$$H(Q,E) = F(E), \qquad \Omega = dE \wedge dQ$$

Equations of motion:

$$\dot{Q} = \frac{\partial H}{\partial E} = \omega(E) = F'(E) = 1 + kE, \qquad \dot{E} = -\frac{\partial H}{\partial Q} = 0$$

- As before, rotation along the energy curves, but now off phase (k=5)!



Phase space mixing: 1D model

- More generally, consider the evolution of a collisionless gas in a 1D potential V(x)

Liouville equation:

$$\frac{\partial f}{\partial t} + \{H, f\} = 0, \qquad H(x,p) = \frac{1}{2}p^2 + V(x)$$

- Assume that $V \in C^\infty(a,b)$ and $V(x) \to \infty$ for $x \to a, b$ and has a unique minimum

- Consider observable associated with smooth test function $\varphi \in C_0^\infty(\Gamma),\ \Gamma := (a,b) \times \mathbb{R}$

$$N_\varphi(t) := \int_\Gamma f(t,x,p)\varphi(x,p)dxdp$$

Theorem

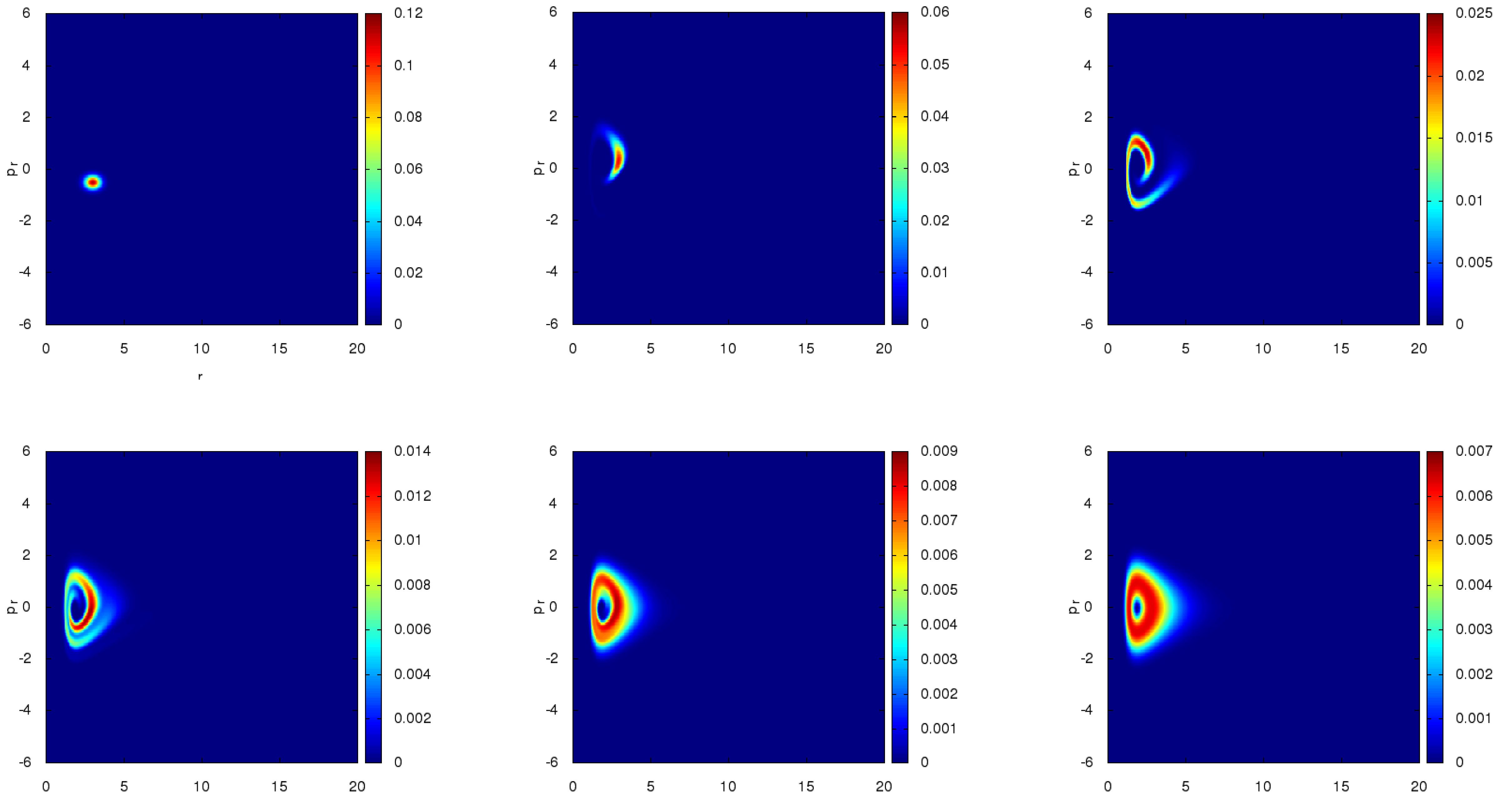
Let f be the unique solution of the Liouville equation for some given initial data $f_0 \in L^1(\Gamma)$. Suppose that for E > 0 the non-degeneracy condition holds: $\frac{d\omega(E)}{dE} \neq 0$

Then,

$$\lim_{t\to\infty} N_\varphi(t) = \int_\Gamma \langle f_0 \rangle_E(x,p)\varphi(x,p)dxdp$$

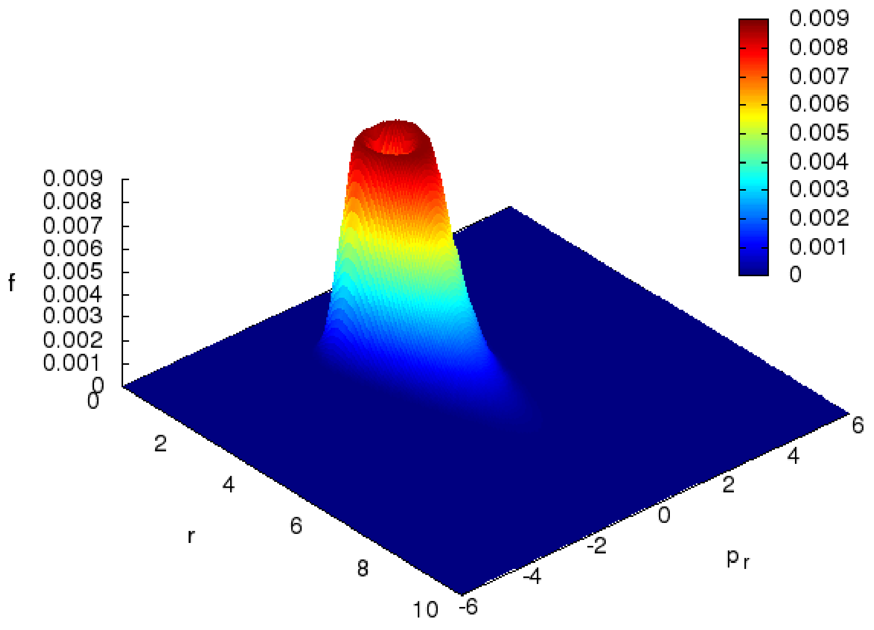
where $\langle f_0 \rangle_E$ denotes the average of f_0 over the energy curve E.

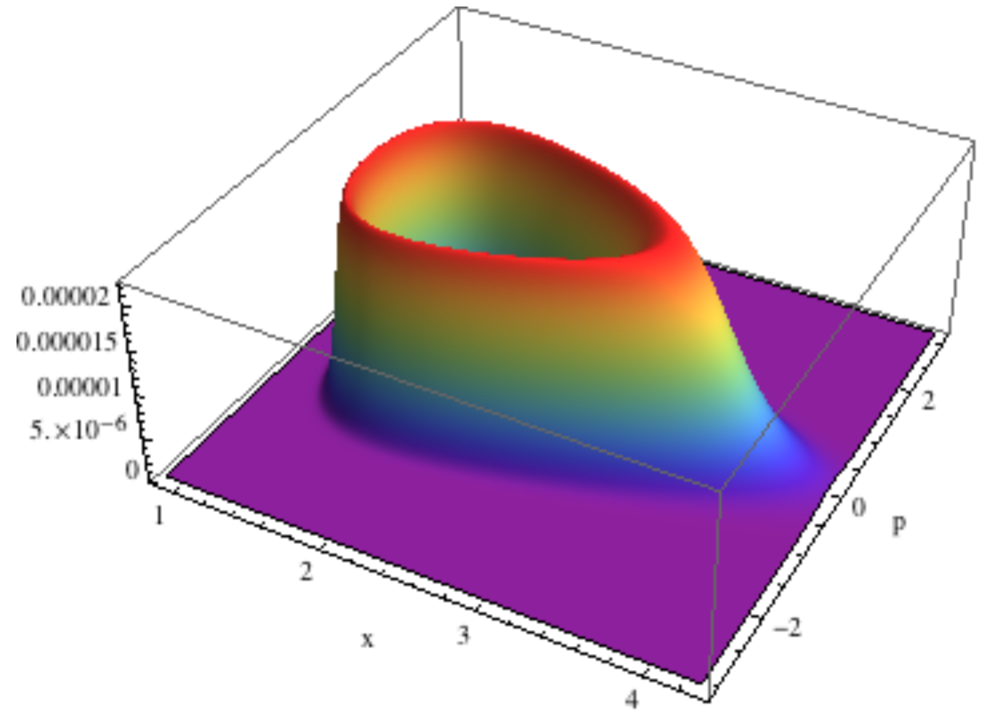
Phase space mixing: dark matter halos



Numerical simulation by Domínguez, Jiménez, Alcubierre, Montoya, Núñez of a Newtonian kinetic gas in an external NFW potential with fixed total angular momentum: Snapshots of the distribution function at different times illustrate the mixing phenomena and the relaxation to a “virialized” state.

Phase space mixing: dark matter halos



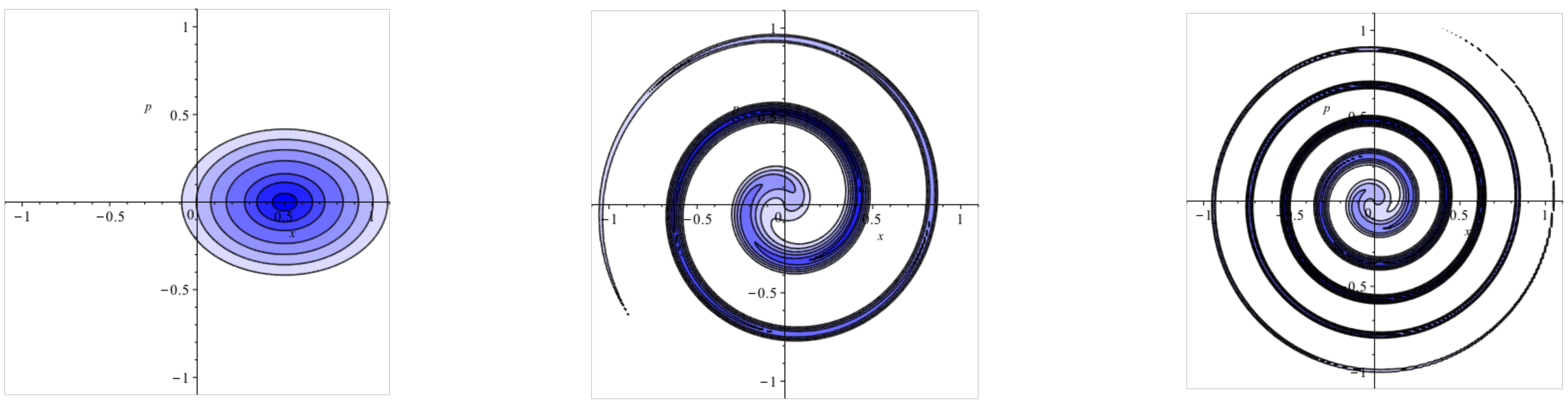


Final distribution function
Left panel: From numerical simulation by Domínguez et al.
Right panel: From a mathematica script by P. Rioseco, averaging the initial distribution function over the energy curves.

Conclusions and Outlook

- Collisionless gas model on rotating black hole background can be integrated explicitly, due to fact that geodesic motion is described by an integrable Hamiltonian system. Thus, the behaviour of spacetime observables (current density, stress-energy tensor) can be analyzed "by inspection" (without solving PDEs).

- Unbounded trajectories lead to dispersion: Ingoing particles are either absorbed by the black hole or are scattered to infinity. Distribution of gas particles on bounded trajectories seem to relax to a steady-state, due to phase-space mixing (conjecture!)



- The use of "good" symplectic coordinates is expected to be useful for the study of perturbed systems, for example when taking into account collisions or the self-gravity of the gas at the perturbative level.

www.redtematicaanyog.mx

Symmetries

- If X is the infinitesimal generator of a one-parameter family of diffeomorphisms on M, there is a natural lift (the complete lift) on T*M given explicitly by

$$\hat{X}^{\mu} = X^{\mu}\frac{\partial}{\partial x^{\mu}} - p_{\alpha}\frac{\partial X^{\alpha}}{\partial x^{\mu}}\frac{\partial}{\partial p_{\mu}}$$

- One can show that this lifted field is the infinitesimal generator of a canonical transformation on T*M; more precisely it is the Hamiltonian vector field associated with the function

$$F(x,p) = p\left(\left.X\right|_{x}\right)$$

- Moreover, for m > 0 the lifted field $\hat{X}$ is everywhere tangent to the corresponding mass shell if and only if X is a Killing vector field (KVF) on (M,g).

- If X is a KVF, then $\hat{X}$ commutes with L.

- If G is a group of isometries of (M,g) generated by KVFs $X_1, X_2, \ldots, X_m$, then we define f to be G-invariant if

$$\hat{X}_a[f] = 0, \quad a = 1, 2, \ldots, m.$$