Cosmological Perturbation Theory
Lecture 2

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Part III: Evolution of Perturbations

Initial Conditions

- At large scales perturbations of components can be understood as patches with a shifted time.

$$\delta Q(x,\eta) \equiv Q(x,\eta) - \bar{Q}(\eta) = Q(\eta + \delta\eta(x)) - \bar{Q}(\eta) \approx \bar{Q}'\delta\eta\,.$$

  Assumption Valid for $k \ll \mathcal{H}$ because of causal contact.

- Adiabatic condition from the conservation of each component.

$$\frac{\delta\rho^{(i)}}{\bar{\rho}^{(i)\prime}} = -\frac{1}{3\mathcal{H}}\frac{\delta\rho^{(i)}}{\bar{\rho}^{(i)} + \bar{P}^{(i)}} = \delta\eta$$

- Then each component of the set of ingredients presents an adiabatic evolution if

$$\bar{P}^{(i)} = \bar{P}^{(i)}\left(\bar{\rho}^{(i)}(x^\mu)\right) \qquad \Rightarrow \qquad -3\mathcal{H}\delta\eta = \frac{\delta^{(\gamma)}}{4/3} = \frac{\delta^{(\nu)}}{4/3} = \delta^{(m)} = \delta^{(b)}$$

- Therefore each component is determined by the total $\delta = \delta\rho/\bar{\rho}$.

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Radiation domination

Tight coupling of baryons and photons at early times before radiation domination due to Thompson scattering.

- The entropy density is given by the relativistic species number density $s \approx g_s n_\gamma$
- The perturbed baryon-entropy ratio is

$$S_b = \delta\left(\frac{n_b}{s}\right)/\frac{n_b}{s} = \frac{\delta(n_b/n_\gamma)}{n_b/n_\gamma} = \frac{\delta n_b}{n_b} - \frac{\delta n_\gamma}{n_\gamma} = \frac{\delta\rho_b}{\bar{\rho}_b} - \frac{3}{4}\frac{\delta\rho_\gamma}{\bar{\rho}_\gamma}$$

  zero under the adiabatic condition $\Rightarrow$ no anisotropic stress.
- Adopting adiabatic initial condition for δ, then $\zeta = \mathrm{const.}$
- Ignoring anisotropic stress (neutrino perturbation) and before horizon entry $\Phi = \Psi$ and,

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- The perturbed matter-(reference fluid) ratio is

$$S_{mf} = \frac{\delta n_m}{n_m} - \frac{\delta n_f}{n_f} = \frac{\delta \rho_m}{\bar{\rho}_m + \bar{P}_m} - \frac{3}{4}\frac{\delta \rho_f}{\bar{\rho}_f + \bar{P}_f}$$

  can be non-zero if non-adiabatic modes exist: Entropy Modes ($N - 1$ for N components)
- Same with the isocurvature modes. In any case need more than one degree of freedom (not present in single field inflation)
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Matter perturbation

1. During matter domination (consider only CDM) The *ij* Einstein Equations shows

$$\Phi_k'' + 6\eta^{-1}\Phi_k{}' = 0.$$

2. Growing mode $\Phi_k = const. = \frac{3}{5}\zeta_k$ (same when baryons contribute).
3. Some scales are inside the horizon at photon decoupling. Distinguish them through transfer function (account for evolution of scales)

$$\delta^{(m)} = -2\Phi_k = -\frac{6}{5}T(k)\zeta_k.$$

4. Inside the horizon, at radiation domination photon density contrast decays due to damping of baryon-photon fluid. Then Φ decays and

$$\delta_k^{(m)\prime\prime} + aH\delta_k^{(m)\prime} = 0, \quad \Rightarrow \quad \delta^{(m)} = \zeta \log(k\eta),$$

which holds until equality when $\delta^{(m)} = \frac{3}{5}\zeta \log(k\eta_{\rm eq})$.

5. This is related to Φ via the Poisson Eqn. and yields the transfer function as

$$\boxed{T(k) = \frac{3}{5}\frac{k_{eq}^2}{k^2}\log\frac{k}{k_{eq}}} \qquad \text{for } k > k_{eq}$$

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Random Fields

Fluctuations are statistically measured as correlation functions or average values.

1. Linear perturbations preserve the shape of the probability distribution
2. Gaussian distribution as estipulated by independent realizations, this happens in quantum perturbations from the inflaton field.
3. The two point correlation contains all info of a Gaussian field

$$< A^*(x)A(x') > = \xi(t, |x - x'|)$$
$$< A^*_k A'_k > = \xi(t, |k - k'|) = \delta_D(|k - k'|)\,\textbf{(homogeneity)} \times P_A(k)\textbf{(isotropy)}$$

$P_A(k)$ is the powerspectrum associated to A.
The *dimensionless* Powerspectrum is $k^3/2\pi^2 k^3 P_A(k) = \mathcal{P}_A(k)$

4. The Transfer function relates A to a single adiabatic **Primordial Perturbation**: $A(k, t) = T_A(k, t, t_0)A(k, t)$

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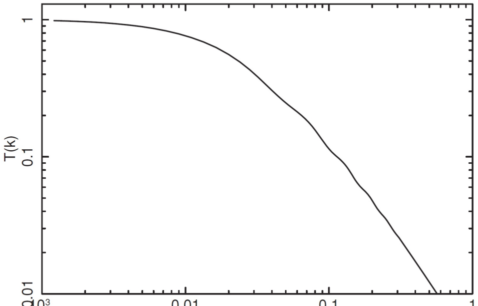

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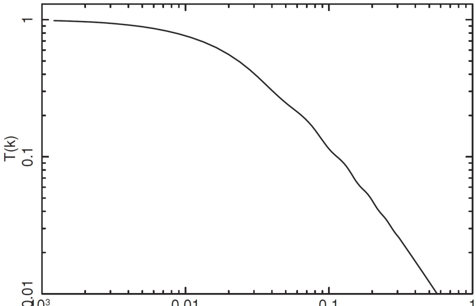

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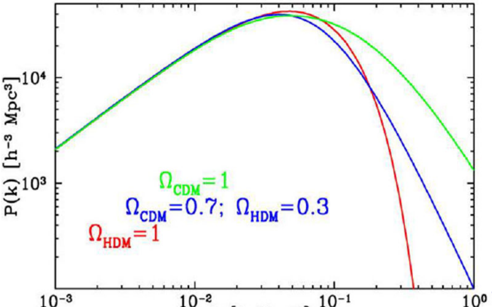

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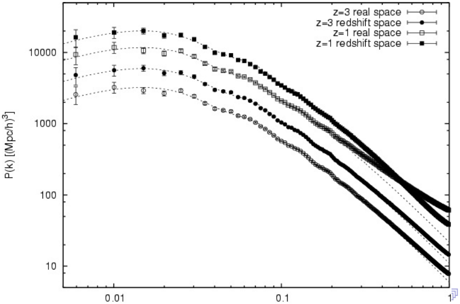


Inflationary conditions

primordial perturbations are quantum fluctuations of inflaton.

- Work with a rescaling of the Mukhanov-Sasaki variable $\mu_S = -2a\sqrt{\gamma}\zeta$ and evolution equation

$$\mu_S'' + \left[k^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}}\right] = 0$$

with $\gamma = 1 - \mathcal{H}'/\mathcal{H}^2$ and $U = \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}}$ the effective scale of instability.

- if $k^2 \gg U$ the solutions are harmonic oscillators
- if $k^2 \ll U$, there is a growing and decaying solution.
- In the intermediate stage and in a slow-roll regime ($\epsilon \ll 1, \quad \eta \ll 1$), solutions are:

$$\mu_S = \sqrt{k\eta}\,[B_1(k)J_\nu(k\eta) - B_2(k)J_{-\nu}(k\eta).]$$

with orders $\nu = -3/2 - 2\epsilon + \eta$, just deviating from scale invariant case $\nu = -3/2$

- The slow roll parameters in single field inflation:

$$\epsilon \equiv \frac{M^2}{2}\left(\frac{V'}{V}\right)^2, \quad \eta \equiv M^2\frac{V''}{V}, \quad \xi_2 \equiv M^4\frac{V'V'''}{V^2}. \tag{1}$$

Inflationary Powerspectrum

Primordial powerspectrum from inflationary solutions $P_\zeta = \frac{1}{8\pi^2}\left|\frac{\mu_S}{a\sqrt{\gamma}}\right|^2$

- Parametrised by amplitude and spectral index

$$\mathcal{P}_\zeta = k^3 P_\zeta(k) = A_s^* \left(\frac{k}{k^*}\right)^{n_s - 1}$$

- Amplitude from inflationary scale and tensors

$$\mathcal{P}_\zeta \simeq \frac{GH^2}{\pi\epsilon}\left[1 - 3\epsilon - 3/2\eta\right] = \frac{1}{24\pi^2 M_{\mathrm{Pl}}^4}\frac{V}{\epsilon}$$

- Spectral index,

$$n_s = 1 - 6\epsilon + 2\eta$$

- Running and running of running,

$$\begin{aligned} n_s &= 1 + 2\eta - 6\epsilon, n_{sk} = \frac{dn_s}{d\ln k} = 16\epsilon\eta - 24\epsilon^2 - 2\xi_2, \\ n_{skk} &= \frac{d^2 n_s}{d\ln k^2} = -192\epsilon^3 + 192\epsilon^2\eta - 32\epsilon\eta^2 - 24\epsilon\xi_2 + 2\eta\xi_2 + 2\xi_3, \end{aligned}$$

- The same analysis for tensors leads to the Gravitational Wave

$$\mathcal{P}_h \simeq \frac{16GH^2}{\pi}\left[1 - 3\epsilon - 2\epsilon\log\frac{k}{k^*}\right] = \frac{1}{24\pi^2 M_{\mathrm{Pl}}^4}\frac{V}{\epsilon} \quad r \equiv \frac{\mathcal{P}_\zeta}{\mathcal{P}_\zeta} = 16\epsilon$$

Observations of CMB

Temperature anisotropies observed today must account for all effects on photon trajectory

- Anisotropies in temperature at the direction of observation $\hat{n}$:

$$\delta T/T(x^\mu, \hat{n}) = \Theta = (\Theta + \Psi)_{dec} + \hat{n}_i v_b^i + \int_{\eta_{dec}}^{\eta_0} d\eta(\phi' + \psi').$$

- At large scale we see super-Hubble scales with negligible Doppler and ISW effect

$$\delta T/T(x^\mu, \hat{n}) = \Theta \approx (\Theta + \Psi)_{dec} = \frac{1}{4}\delta^{(\gamma)} + \Psi = \frac{1}{4}\frac{4}{3}(-2)\Psi + \Psi = \frac{1}{3}\Psi.$$

  This shows the amplitude of the primordial anisotropies. Also manifest in the two-point correlation of $\Theta(\hat{n}) = \sum_{l,m} a_{l,m} Y_{l,m}(\hat{n})$

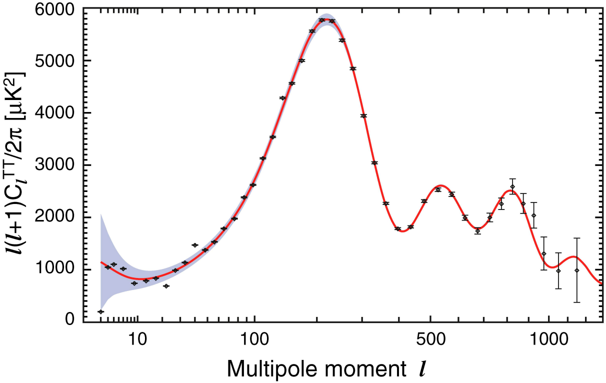
$$\langle \Theta(\hat{n})\Theta^*(\hat{n}') \rangle = \langle a_{lm} a_{l'm'}* \rangle = \delta_{ll'}\delta_{mm'} \left[\frac{1}{2\pi^2} \int \frac{dk}{k} \Theta_l^2(\eta_0, k) \mathcal{P}_\zeta(k) \right] = \delta_{ll'}\delta_{mm'} C_l.$$

- In practice we do not have many realizations and

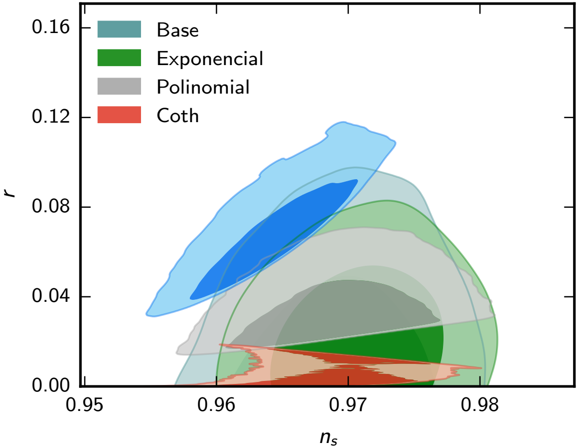
$$C_l^{obs} \neq C_l = \frac{1}{2l+1} \sum_{-l<m<l} |a_{lm}|^2$$

- This leads to cosmic variance $\langle (C_l^{obs} - C_l)^2 \rangle = \frac{2}{2l+1} C_l$

CMB multipole spectrum



CMB parameter estimation



N. Barbosa et al. **arXiv:1711.06693**

Outline Part IV

- From Boltzmann equation to Fluid variables
- Solutions to the Perturbation Equations
- Zel'dovich approximation
- 2 Point Correlation Function
- Powerspectrum
- Separate unvierses and Spherical Collapse

Elements for description of Structure formation

- Scales below the Hubble horizon $r_H = c/H$ or wavenumbers below the comoving horizon $\mathcal{H}^{-1}$

$$k/aH = k/\mathcal{H} \approx k\eta \ll 1,$$

- Dark Matter domination

$$\begin{aligned} \langle u^i \rangle &= u^i \quad \text{No dispersion} \\ c_{\mathrm{CDM}\gamma} &= 0 \quad \text{No interactions} \\ P &= 0 \quad \text{Presureless fluid} \end{aligned}$$

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The Boltzmann Equation

- Set of N Colisionless particles represented by the phase distribution funciton

$$f(\mathbf{x},\mathbf{p},\eta) = \sum_n \delta_{\rm D}(\mathbf{x}-\mathbf{x}_n(\eta))\delta_{\rm D}(\mathbf{p}-\mathbf{p}_n(\eta)),$$

  defines the density

$$\rho(\mathbf{x},\eta) = ma^{-3}\int d^3pf,$$

  the local mean velocity

$$\rho\langle u^i\rangle_p(\mathbf{x},\eta) = a^{-4}\int d^3pp^if,$$

  and the stress tensor from the second momentum of the distribution

$$\rho\langle u^iu^j\rangle_p = a^{-5}\int d^3p\frac{1}{m}p^ip^jf = \rho v^iv^j + \rho\sigma^{ij}.$$

- Evolution from Vlasov or Colisionless Boltzmann Equation

$$\frac{df}{d\eta}(\mathbf{x},\mathbf{p},\eta) = \frac{\partial f}{\partial\eta} + \frac{1}{am}\mathbf{p}\cdot\frac{\partial f}{\partial\mathbf{x}} - am\frac{\partial\phi}{\partial\mathbf{x}}\cdot\frac{\partial f}{\partial\mathbf{p}} = 0,$$

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- The Newtonian field equation or Poisson equation for the potential Φ:

$$\nabla^2\Phi = 4\pi G a^2 \rho.$$

Moments of the Boltzmann equation yield the hydrodynamics governing equations.

$$\begin{aligned}\partial_\eta \rho(\mathbf{x},\eta) &= -\nabla\cdot(\rho\mathbf{u})\\ (\partial_\eta + \mathbf{u}\cdot\nabla)\,\mathbf{u} &= -\nabla\Phi - \frac{1}{\rho}\partial_j\left(\rho\sigma^{ij}\right).\end{aligned}$$

For the background, $\mathbf{u} = \mathbf{x}\mathcal{H}$ and the **inhomogeneous** potential ϕ_0 yield

$$\begin{aligned}\partial_t\bar{\rho} &= -3\mathcal{H}\left(\bar{\rho}\right),\\ \partial_t\mathcal{H} &= -\frac{4}{3}\pi G a^2\bar{\rho},\end{aligned}$$

with solutions $\bar{\rho} = \bar{\rho}a^{-3}$ and $a \propto \eta^2$

Perturbation Equations

For density perturbation $\delta = \delta\rho/\bar{\rho}$ and peculiar velocities $\mathbf{v} = \mathbf{u} - \mathcal{H}\mathbf{x}$,

$$\begin{aligned} \partial_\eta \delta(\mathbf{x}, \eta) + \partial_j((1+\delta)v^j) &= 0, \\ \partial_\eta v^i(\mathbf{x}, \eta) + v^j \partial_j v^i + \mathcal{H} v^i + \partial^i \phi &= -\frac{1}{\rho}\partial_j(\rho\sigma^{ij}). \end{aligned}$$

and the Poisson equation for the perturbed potential ϕ is

$$\nabla^2 \phi = 4\pi G \bar{\rho} \delta.$$

Combining the above we arrive at the evolution equation for density perturbations **and truncating the Boltzmann hiearchy at second order** we get, in the linear limit,

$$\begin{aligned} \partial_\eta \delta^{(1)} &= -\partial_j v^{(1)j} \equiv \theta^{(1)}, \\ \partial_\eta v^{(1)i} + \mathcal{H} v^{(1)i} &= -\partial^i \phi^{(1)}. \end{aligned}$$

Linearizing the last term and combining both Eqs.

$$\delta^{(1)\prime\prime} + \mathcal{H}\delta^{(1)\prime} - \frac{3}{2}\mathcal{H}^2 \Omega_m \delta^{(1)} = 0 \qquad (2)$$

The Newtonian Regime from GR

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \gamma_{ij}(\mathbf{x},\eta)dx^i dx^j\right],$$

The deformation Tensor:

$$\vartheta^\mu_\nu \equiv a u^\mu{}_{;\nu} - \mathcal{H}\delta^\mu_\nu \quad \rightarrow \vartheta^i_j = -K^i_j = \gamma^{ik}\gamma'_{jk},$$

Relativistic Equations in the Synchronous gauge

$$\begin{aligned} \delta' + \vartheta(1+\delta) &= 0, \\ \vartheta' + \mathcal{H}\vartheta + \vartheta^i_j\vartheta^j_i + 4\pi G a^2\bar{\rho}\delta &= 0, \\ \vartheta^2 - \vartheta^i_j\vartheta^j_i + 4\mathcal{H}\vartheta + \mathcal{R} &= 16\pi G a^2\bar{\rho}\delta, \end{aligned}$$

Dictionary of Newtonian Vs. Relativistic (at non-linear order)

$$\begin{aligned} \mathcal{R} &\rightarrow \nabla^2\Phi_N \\ \delta_c &\rightarrow \delta_N \\ \vartheta^i_j &\rightarrow \nabla^i\nabla_j v_N \end{aligned}$$

* This is valid at all scales and all orders and at non-linear level except for the constrain equation (3)

Solution to linear equations

- In cosmic time t the evolution equation is

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}\delta = 0$$

- Two Solutions:

$$\delta^{(1)}(t, \mathbf{x}) = \delta_+^{(1)}(\mathbf{x})D_+(t) + \delta_-^{(1)}(\mathbf{x})D_-(t)\,. \tag{3}$$

- Identifying equations we see that $D_- = H$.
- Using the Wronskian or a particular solution

$$D_+(t) = H(t)\int \frac{dt}{aH^2(t)} = H(a)\int \frac{da}{(aH)^3} \quad \to \quad \frac{\mathcal{H}}{a}\int \frac{ds}{(\mathcal{H}(s))^3}$$

- In CDM domination (Einstein-de Sitter Universe)

$$D_+(\eta) = a(\eta) = \eta^2 \tag{4}$$

- In LCDM growth is suppressed wrt E-dS. Define the growth suppression factor

$$f \equiv \frac{d\log\delta^{(1)}}{d\log a} = \frac{D_+'}{\mathcal{H}D_+} = \frac{\theta}{\mathcal{H}\delta}$$

- Defining the growth factor

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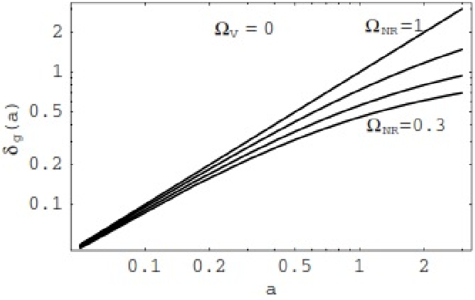
It is evident that in E-dS $f = 1$. In LCDM it is

$$f = -\frac{3}{2}\Omega_m + \Omega_m a(\eta) \frac{1}{\delta_+^{(1)}(\mathbf{x})} = \Omega_m^{\gamma}$$

- The approximate value if $1 - \Omega_m \ll 1$ is

$$\gamma = 6/11$$



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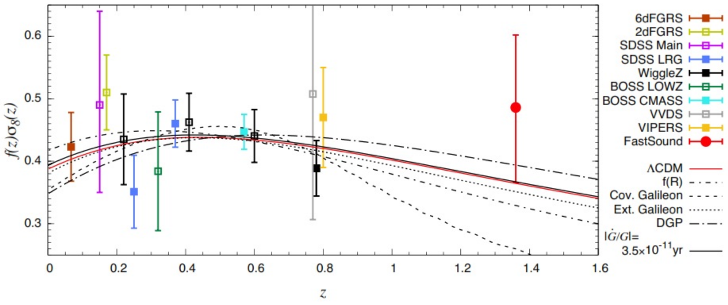

Standard Perturbation Theory

Standard Perturbation Theory

$$\dot{\delta}(\mathbf{k},\eta) + \theta = -\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_D(\mathbf{k}_1+\mathbf{k}_2)\alpha(\mathbf{k}_1,\mathbf{k}_2)\theta(\mathbf{k}_1)\delta(\mathbf{k}_2),$$

$$\dot{\theta}(\mathbf{k},\eta) + \mathcal{H}\theta + \frac{3}{2}\Omega_m\mathcal{H}^2\delta = -\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_D(\mathbf{k}_1+\mathbf{k}_2)\beta(\mathbf{k}_1,\mathbf{k}_2)\theta(\mathbf{k}_1)\theta(\mathbf{k}_2),$$

with

$$\alpha(\mathbf{k}_1,\mathbf{k}_2) = \frac{\mathbf{k}_{12}\cdot\mathbf{k}_1}{k_1^2}, \qquad \beta(\mathbf{k}_1,\mathbf{k}_2) = \frac{k_{12}^2(\mathbf{k}_1\cdot\mathbf{k}_2)}{2k_1^2k_2^2}$$

Formally expand the matter and velocity fields

$$\delta(\mathbf{k},\eta) = \delta^{(1)}(\mathbf{k},\eta) + \delta^{(2)}(\mathbf{k},\eta) + \delta^{(3)}(\mathbf{k},\eta) + \cdots,$$

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$$\delta^{(n)} = D_+^n(\eta)\hat{\delta}^{(n)}(\mathbf{k}) \quad \text{and} \quad \theta^{(n)} = \mathcal{H}(\eta)D_+^n(\eta)\hat{\theta}^{(n)}(\mathbf{k})$$

This is possible for $f = d\ln\delta/d\ln a = \Omega_m^{1/2}$ [Bernardeau, ApJ 433, 1 (1994)]

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Standard Perturbation Theory

Obtain the recurrence relations

$$n\hat{\delta}^{(n)} + \hat{\theta}^{(n)} = A_n, \quad 3\hat{\delta}^{(n)} + (1+2n)\hat{\theta}^{(n)} = B_n,$$

where

$$A_n(\mathbf{k}) = -\int d^3k_1 d^3k_2 \delta_{\rm D}(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})\alpha(\mathbf{k}_1, \mathbf{k}_2) \sum_{m=1}^{n-1} \hat{\theta}^{(m)}(\mathbf{k}_1)\hat{\delta}^{(n-m)}(\mathbf{k}_2),$$

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The inverse relations are

$$\hat{\delta}^{(n)}(\mathbf{k}) = \frac{(1+2n)A_n(\mathbf{k}) - B_n(\mathbf{k})}{(2n+3)(n-1)}, \quad \hat{\theta}^{(n)}(\mathbf{k}) = \frac{-3A_n(\mathbf{k}) + nB_n(\mathbf{k})}{(2n+3)(n-1)}.$$

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Standard Perturbation Theory

$$\delta^{(n)}(\mathbf{k},\eta) = D_+^n(\eta)\int\prod_{m=1}^{n} d^3q_m\,\delta_{\rm D}(\mathbf{q}_1+\cdots+\mathbf{q}_n-\mathbf{k})F_n(\mathbf{q}_1,\ldots,\mathbf{q}_n)\delta_0^{(1)}(\mathbf{q}_1)\cdots\delta_0^{(1)}(\mathbf{q}_n)$$

$$\theta^{(n)}(\mathbf{k},\eta) = -\mathcal{H}^n(\eta)D_+^n(\eta)\int\prod_{m=1}^{n} d^3q_m\,\delta_{\rm D}(\mathbf{q}_1+\cdots+\mathbf{q}_n-\mathbf{k})G_n(\mathbf{q}_1,\ldots,\mathbf{q}_n)\delta_0^{(1)}(\mathbf{q}_1)\cdots\delta_0^{(1)}(\mathbf{q}_n)$$

With kernels F and G

$$F_n(\mathbf{q}_1,\ldots,\mathbf{q}_n) = \sum_{m=1}^{n-1}\frac{G_m(\mathbf{q}_1,\ldots,\mathbf{q}_m)}{(2n+3)(n-1)}\Big[(2n+1)\alpha(\mathbf{k}_1,\mathbf{k}_2)F_{n-m}(q_{m+1},\ldots,q_n) + 2\beta(\mathbf{k}_1,\mathbf{k}_2)G_{n-m}(q_{m+1},\ldots,q_n)\Big]$$

$$G_n(\mathbf{q}_1,\ldots,\mathbf{q}_n) = \sum_{m=1}^{n-1}\frac{G_m(\mathbf{q}_1,\ldots,\mathbf{q}_m)}{(2n+3)(n-1)}\Big[3\alpha(\mathbf{k}_1,\mathbf{k}_2)F_{n-m}(q_{m+1},\ldots,q_n) + 2n\beta(\mathbf{k}_1,\mathbf{k}_2)G_{n-m}(q_{m+1},\ldots,q_n)\Big].$$

$\mathbf{k}_1 \equiv \mathbf{q}_1+\cdots+\mathbf{q}_m$ and $\mathbf{k}_2 \equiv \mathbf{q}_{m+1}+\cdots+\mathbf{q}_n$.

Standard Perturbation Theory at Second Order

For example, the $n = 2$ case gives

$$F_2(\mathbf{q}_1, \mathbf{q}_2) = \frac{5}{7} + \frac{1}{2}\frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1 q_2}\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) + \frac{2}{7}\frac{(\mathbf{q}_1 \cdot \mathbf{q}_2)^2}{q_1^2 q_2^2},$$

$$G_2(\mathbf{q}_1, \mathbf{q}_2) = \frac{3}{7} + \frac{1}{2}\frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1 q_2}\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) + \frac{4}{7}\frac{(\mathbf{q}_1 \cdot \mathbf{q}_2)^2}{q_1^2 q_2^2}$$

Powerspectrum at Second Order

$$
\begin{aligned}
\langle \delta(\mathbf{k})\delta(\mathbf{k}')\rangle' &= \langle (\delta^{(1)}(\mathbf{k}) + \delta^{(2)}(\mathbf{k}) + \delta^{(3)}(\mathbf{k}) + \cdots)(\delta^{(1)}(\mathbf{k}') + \delta^{(2)}(\mathbf{k}') + \delta^{(3)}(\mathbf{k}') + \cdots)\rangle' \\
&= P_L(\mathbf{k},\eta) + 2P^{(13)}(\mathbf{k},\eta) + P^{(22)}(\mathbf{k},\eta) + \cdots \\
&= D_+^2(\eta)P_L(\mathbf{k}) + D_+^4(\eta)(2P^{(13)}(\mathbf{k}) + P^{(22)}(\mathbf{k})) + \cdots
\end{aligned}
$$

$P^{(nm)}(\mathbf{k}) = \langle \delta^{(n)}(\mathbf{k})\delta^{(m)}(\mathbf{k}')\rangle'$

$$
P^{(22)}(k) = 2\int d^3q P_L(q)P_L(|\mathbf{k}-\mathbf{q}|)[F_2^{(s)}(\mathbf{q},\mathbf{k}-\mathbf{q})]^2.
$$

$$
2P^{(13)}(k) = 6P_L(k)\int d^3q P_{11}(q)F_3^{(s)}(\mathbf{q},-\mathbf{q},\mathbf{k}).
$$

Performing the angular integrals

$$
\begin{aligned}
P^{\rm SPT}_{\text{1-loop}} = \exp\left[-\frac{k^6}{6\pi^2}\int dp P_L(p)\right]&\left\{P_L(k) + \frac{1}{98}\frac{k^3}{4\pi^2}\int_0^\infty dr P_L(kr)\right. \\
&\times \int_{-1}^1 dx P_L(k\sqrt{1+r^2-2rx})\frac{(3r+7x-10rx^2)^2}{(1+r^2-2rx)^2} + \frac{1}{252}\frac{k^3}{4\pi^2}P_L(k)\int_0^\infty dr P_L(kr) \\
&\left.\times\left[\frac{12}{r^2} - 158 + 100r^2 - 42r^4 + \frac{3}{r^3}(r^2-1)^3(7r^2+2)\ln\left|\frac{1+r}{1-r}\right|\right]\right\}
\end{aligned}
$$

Powerspectrum at Second Order

$$\begin{aligned}\langle\delta(\mathbf{k})\delta(\mathbf{k}')\rangle' &= \langle(\delta^{(1)}(\mathbf{k})+\delta^{(2)}(\mathbf{k})+\delta^{(3)}(\mathbf{k})+\cdots)(\delta^{(1)}(\mathbf{k}')+\delta^{(2)}(\mathbf{k}')+\delta^{(3)}(\mathbf{k}')+\cdots)\rangle' \\ &= P_L(\mathbf{k},\eta)+2P^{(13)}(\mathbf{k},\eta)+P^{(22)}(\mathbf{k},\eta)+\cdots \\ &= D_+^2(\eta)P_L(\mathbf{k})+D_+^4(\eta)(2P^{(13)}(\mathbf{k})+P^{(22)}(\mathbf{k}))+\cdots\end{aligned}$$

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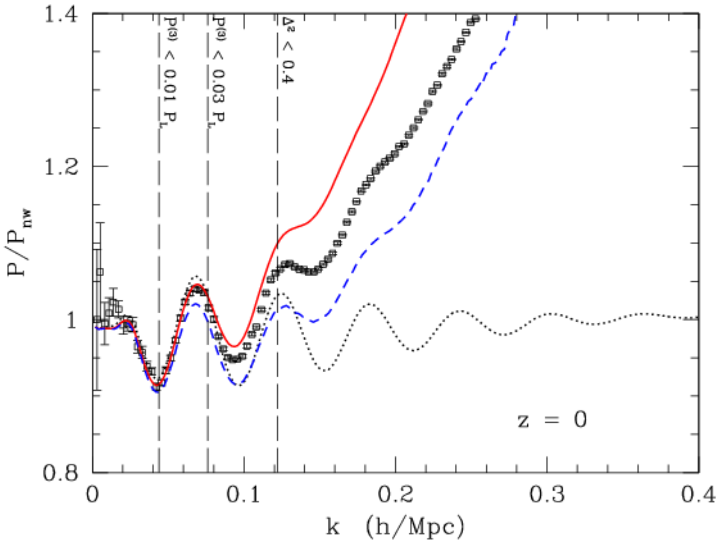
Powerspectrum at Second and Third Order in SPT



Lagrangian Perturbation Theory

LPT: where the coordinates are comoving with fluid particles

$$\mathbf{x}(\mathbf{q},\eta) = \mathbf{q} + \Psi(\mathbf{q},\eta), \qquad \Psi(\mathbf{q},\eta_i) = 0,$$

The Lagrangian displacement $\Psi^i(\mathbf{q},\eta)$, is related to peculiar velocity

$$\dot{\Psi}^i(\mathbf{q},\eta) = v^i.$$

$$\Rightarrow 1+\delta = \frac{\rho_0(\mathbf{q})}{\mathrm{Det}|\delta_{ik} + \partial_i \Psi_k(q,\eta)|}$$

Continuity equation is thus integrated.
Zel'dovich approximation means free streaming (balistic approximation)

$$\mathbf{x} = \mathbf{q} + a(\eta)\mathbf{v}(q),$$

$$\Rightarrow \rho(q,\eta) = \prod_{\ell=1}^{3} \frac{\rho_0}{1 + a\lambda_\ell(q)}$$

with λ_ℓ the eigenvalues of $\partial_i \Psi_j$

Zel'dovich Approximation

Zel'dovich approximation means free streaming

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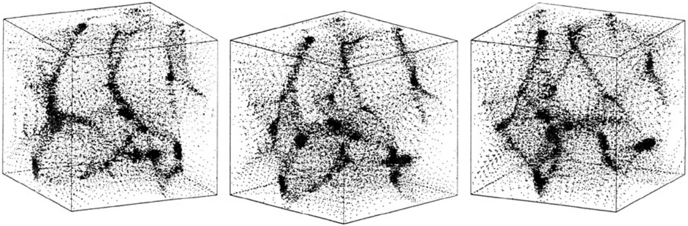

Lagrangian Perturbation Theory

Continuity, Euler and VDT equations $\Longrightarrow$

$$\ddot{\Psi}^i(\mathbf{q},\eta) + \mathcal{H}\dot{\Psi}^i + \partial^i\phi(\mathbf{q}+\Psi) = -\frac{1}{1+\delta}\partial_j((1+\delta)\sigma^{ij}),$$

$$\dot{\sigma}^{ij}(\mathbf{q},\eta) + 2\mathcal{H}\sigma^{ij} + \sigma^{ik}\partial_k\dot{\Psi}^j + \sigma^{jk}\partial_k\dot{\Psi}^i = 0.$$

∇ (∂) denotes derivatives with respect to Lagrangian (Eulerian) coordinates.

- In the following we shall consider the case $\sigma^{ij} = 0$.

Before shell-crossing the conservation of mass ($(1+\delta(\mathbf{x}))d^3x = (1+\delta(\mathbf{q}))d^3q$) implies

$$1+\delta(\mathbf{x}) = \frac{1}{\det(I+\nabla\Psi(\mathbf{q}))} = J^{-1}.$$

where J is the determinant of the Jacobian matrix of the transformation from Eulerian to Lagrangian coordinates: $J^i_j \equiv \delta^i_j + \nabla_j\Psi^i$. Alternatively we can rewrite the relation as

$$1+\delta(\mathbf{x}) = \int \delta_D(\mathbf{x}-\mathbf{q}-\Psi(\mathbf{q},\eta))d^3q = \int d^3q\frac{d^3k'}{(2\pi)^3}e^{i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{q}-\Psi(\mathbf{q},\eta))}$$

$$\delta(\mathbf{k}) = \int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}}(e^{-i\mathbf{k}\cdot\Psi(\mathbf{q},\eta)} - 1)$$

Lagrangian Perturbation Theory

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Lagrangian Recurrence relations

$$\ddot{\Psi}^i(\mathbf{q},\eta) + \mathcal{H}\dot{\Psi}^i + \partial^i\phi(\mathbf{q}+\Psi) = 0$$

Solving order by order:

$$\Psi^{(n)i}(\mathbf{k},\eta) = i\frac{D_+^n}{n!}\int_{\mathbf{k}} L^{(n)i}(\mathbf{k}_1,\ldots,\mathbf{k}_n)\delta_0^{(1)}(\mathbf{k}_1)\cdots\delta_0^{(1)}(\mathbf{k}_n),$$

The first three terms are [Bouchet et al. A&A 296, 575 (1995)]

$$L^{(1)}(\mathbf{k}) = \frac{\mathbf{k}}{k^2}, \qquad L^{(2)}(\mathbf{k}_1,\mathbf{k}_2) = \frac{3}{7}\frac{\mathbf{k}}{k^2}\left[1-\left(\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1k_2}\right)^2\right],$$

$$L^{(3)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{5}{7}\frac{\mathbf{k}}{k^2}\left[1-\left(\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1k_2}\right)^2\right]\left[1-\left(\frac{(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{k}_3}{|\mathbf{k}_1+\mathbf{k}_2|k_3}\right)^2\right]$$
$$-\frac{1}{3}\frac{\mathbf{k}}{k^2}\left[1-3\left(\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1k_2}\right)^2+2\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_2\cdot\mathbf{k}_3)(\mathbf{k}_3\cdot\mathbf{k}_1)}{k_1^2k_2^2k_3^2}\right],$$

- Recurrence relation were unknown until recently. Matsubara

Lagrangian Powerspectrum

$$\langle\Big(\overbrace{\int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}} + \delta(\mathbf{k})}^{\int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}} e^{-i\mathbf{k}\cdot\Psi}}\Big)\Big(\int d^3q' e^{-i\mathbf{k}'\cdot\mathbf{q}'} + \delta(\mathbf{k}')\Big)\rangle =$$

$$\langle\delta(\mathbf{k})\delta(\mathbf{k}')\rangle + \langle\int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}} \int d^3q' e^{-i\mathbf{k}'\cdot\mathbf{q}'}\rangle + \underbrace{\langle\delta(\mathbf{k})\int d^3q' e^{-i\mathbf{k}'\cdot\mathbf{q}'}\rangle + \langle\delta(\mathbf{k}')\int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}}\rangle}_{=0}.$$

$$\langle\delta(\mathbf{k})\delta(\mathbf{k}')\rangle = -(2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}')\int d^3q e^{i\mathbf{q}\cdot\mathbf{k}} + \int d^3q_1 d^3q_2 e^{-i\mathbf{k}\cdot\mathbf{q}_1} e^{-i\mathbf{k}'\cdot\mathbf{q}_2}\langle e^{-i\mathbf{k}\cdot\Psi(\mathbf{q}_1)-i\mathbf{k}'\cdot\Psi(\mathbf{q}_2)}\rangle. \quad (*)$$

Def: $\vec{Q} \equiv (\mathbf{q}_1+\mathbf{q}_2)/2\ ,\ \mathbf{q} \equiv \mathbf{q}_2 - \mathbf{q}_1 \quad \Rightarrow \quad \mathbf{q}_1 = (2\vec{Q}-\mathbf{q})/2\ ,\ \mathbf{q}_2 = (2\vec{Q}+\mathbf{q})/2.$

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Lagrangian Powerspectrum

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The matter power spectrum expressed in terms of displacement fields is ([Taylor & Hamilton, MNRAS 282, 767 (1996)]),

$$P_{\rm LPT}(\mathbf{k}) = \int d^3q e^{i\mathbf{k}\cdot\mathbf{q}}(\langle e^{i\mathbf{k}\cdot\Delta}\rangle - 1).$$

where $\mathbf{q} = \mathbf{q}_2 - \mathbf{q}_1$ is the Lagrangian coordinates separation and

$$\Delta(\mathbf{q}) \equiv \Psi(\mathbf{q}_2, \eta) - \Psi(\mathbf{q}_1, \eta).$$

Now, we can use the cumulant expansion theorem,

$$\langle e^{iX}\rangle = \exp\left(\sum_{N=1}^{\infty}\frac{i^N}{N!}\langle X^N\rangle_c\right),$$

to rewrite the PS as

$$(2\pi)^3\delta_{\rm D}(\mathbf{k}) + P^{\rm LPT}(\mathbf{k}) = \int d^3q e^{i\mathbf{k}\cdot\mathbf{q}}\exp\left[-\frac{1}{2}k_ik_j\langle\Delta^i\Delta^j\rangle_c - \frac{i}{6}k_ik_jk_k\langle\Delta^i\Delta^j\Delta^k\rangle_c + \cdots\right].$$

Different expansions of the exponential lead to different (resummation) Schemes

- iPT (Matsubara formalism): Keeps in the exponential terms evaluated at $\mathbf{q} = 0$
- CLPT: Expands terms that goes to zero as $q \to \infty$
- ...see Vlah, Seljak, and Baldauf [arXiv:1410.1617]

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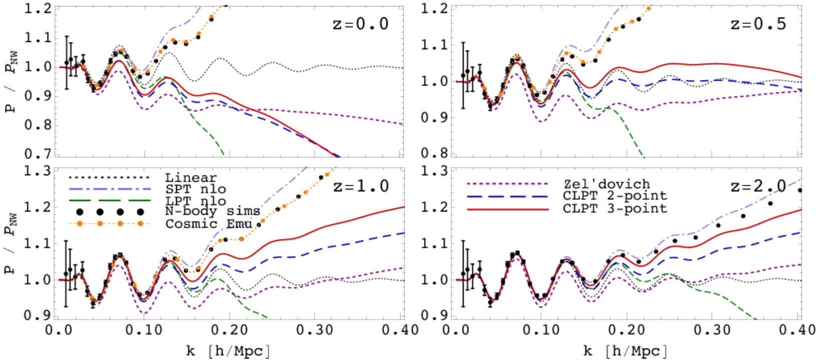
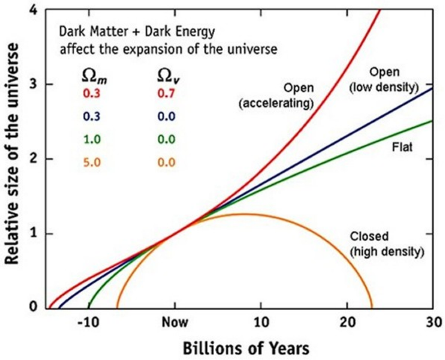
Powerspectrum at Second Order in LPT



Spherical Collapse

0) Different choices for a background universe.

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\bar{\rho} + \frac{1}{3}\Lambda.$$



Spherical Collapse

0) Expansion for a background universe.

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\bar{\rho}$$

1) Consider an overdensity with uniform density $\rho_a = \bar{\rho} + \delta\rho$ above the cosmological horizon $1/\mathcal{H}$ and with scale factor $R(t)$, on top:

$$H_a^2 = \frac{8\pi G}{3}\rho_b = \frac{8\pi G}{3}\left[\rho_b + \delta\rho - \delta\rho\right] = \frac{8\pi G}{3}\left[\rho_a - \delta\rho\right]$$

2) For a common initial expansion $H_a(t_i) = H_b(t_i)$ the overdensity is interpreted as positive curvature.

$$\frac{\kappa c^2}{R^2(t_i)} = \frac{8\pi G}{3}\delta\rho(t_i).$$

3) Deriving the Friedmann equation we have a Keppler problem because there is no mass transfer

$$\frac{d^2 R}{dt^2} = -GM/R^2.$$

4) And the solution in terms of the conformal time:

$$R(\eta) = \frac{R_{max}}{2}\left(1 - \cos\eta\right),$$
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Spherical Collapse

5) There is a maximum Radius for collapse

$$R_{\rm max} = \frac{\Omega_a(t_i)}{\Omega_a(t_i) - 1},$$

6) If we take small times and expand around $\eta = 0$,

$$\begin{aligned}
R &= R_{max}\left(\frac{1}{4}\eta^2 - \frac{1}{48}\eta^4\right), \\
t &= \frac{1}{\pi}t_{max}\left(\frac{1}{6}\eta^3 - \frac{1}{120}\eta^5\right). \\
R(t) &= \frac{1}{4}R_{max}\left(6\pi\frac{t}{t_{max}}\right)^{2/3}\left[1 - \frac{1}{20}\left(6\pi\frac{t}{t_{max}}\right)^{2/3}\right],
\end{aligned}$$

7) Take the ratio of densities $\rho_a/\rho_b = 1 + \delta$:

$$1 + \delta{\rm lin} = \frac{\rho_a}{\rho_b} = \frac{a^3}{R^3} = 1 + \frac{3}{20}\left(6\pi\frac{t}{t_{max}}\right)^{2/3},$$

Spherical Collapse

8) In the non-linear regime, the density diverges, while the linear density

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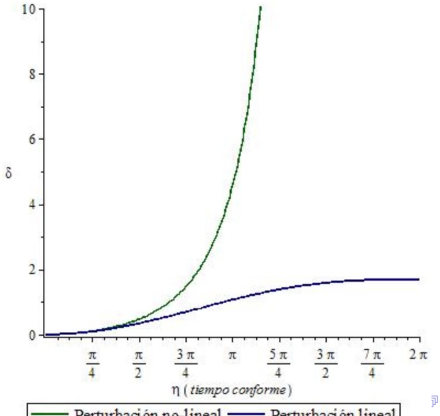
$$\delta_{\text{lin,t.a.}} = \frac{3}{20}(6\pi)^{2/3} = 1{,}06, \qquad \delta_{\text{lin,col}} = \frac{3}{20}(12\pi)^{2/3} = 1{,}686$$

9) **Spherical collapse guides us to detachment and virialization criteria.**

$$\delta_{\text{nl,vir}} = \delta(R = R_{\text{max}}/2) \simeq 178$$



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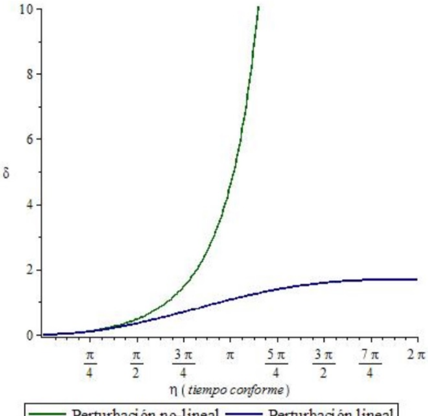

Cosmological Perturbation Theory

That's all folks!
(for now)

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