Gravity: Newtonian, post-Newtonian, Relativistic

X Mexican School on Gravitation & Mathematical Physics

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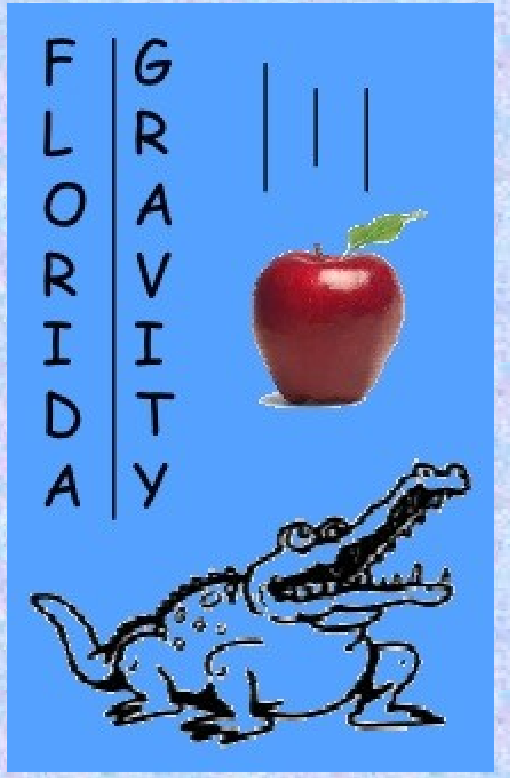
Clifford Will
Distinguished Professor of Physics
University of Florida
Chercheur Associé
Institut d'Astrophysique de Paris

http://phys.ufl.edu/~cmw/
cmw@physics.ufl.edu



Outline of the Lectures*

Part 1: Newtonian Gravity

- Foundations
- Equations of hydrodynamics
- Spherical and nearly spherical bodies
- Motion of extended fluid bodies

Part 2: Newtonian Celestial Mechanics

- Two-body Kepler problem
- Perturbed Kepler problem

*Based on *Gravity: Newtonian, post-Newtonian, General Relativistic,* by Eric Poisson and Clifford Will (Cambridge U Press, 2014)

Outline of the Lectures*

Part 3: General Relativity

- Einstein equivalence principle
- GR field equations

Part 4: Post-Newtonian & post-Minkowskian theor

- Formulation
- Near-zone physics
- Wave-zone physics
- Radiation reaction

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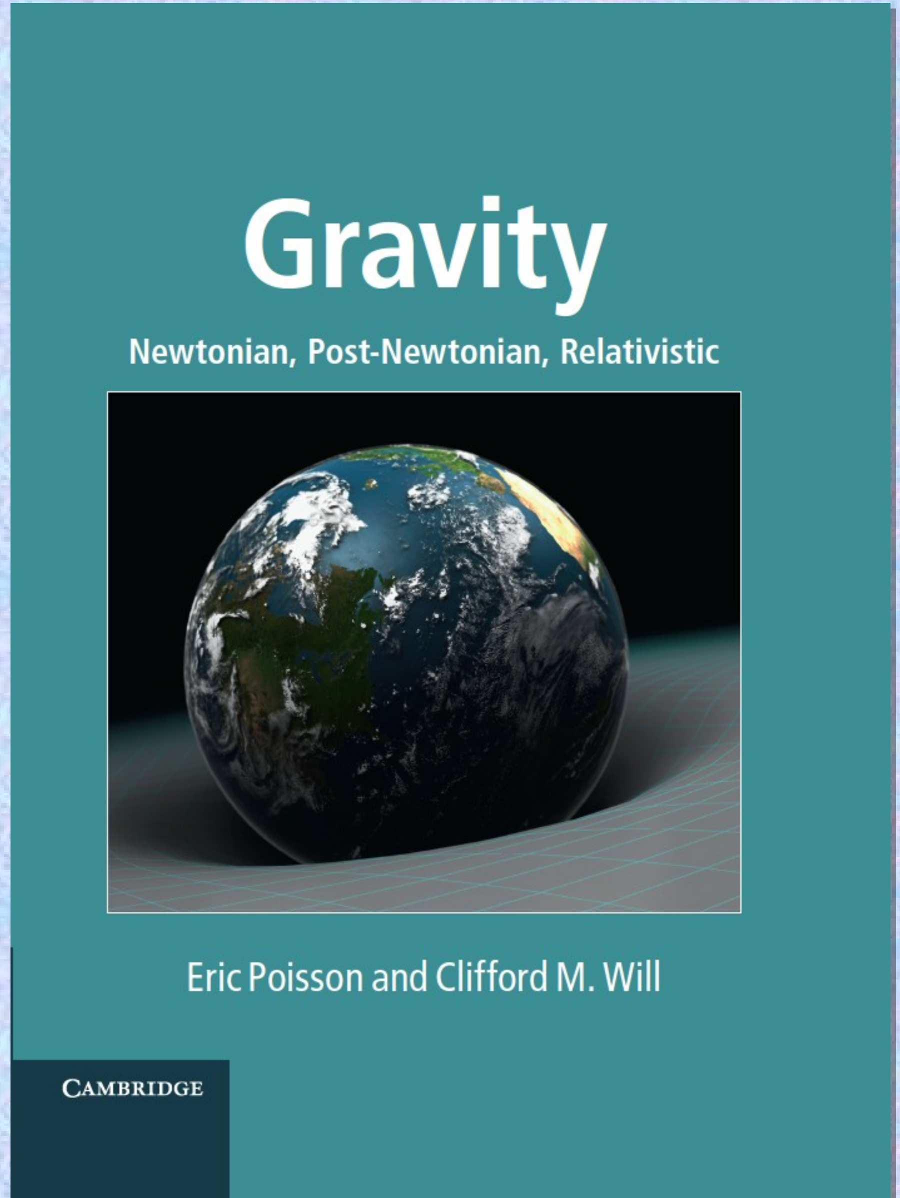

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Foundations of Newtonian Gravity

Newton's 2nd law and the law of gravitation:

$$m_I \boldsymbol{a} = \boldsymbol{F}$$

$$\boldsymbol{F} = -G m_G M \boldsymbol{r}/r^3$$

The principle of equivalence:

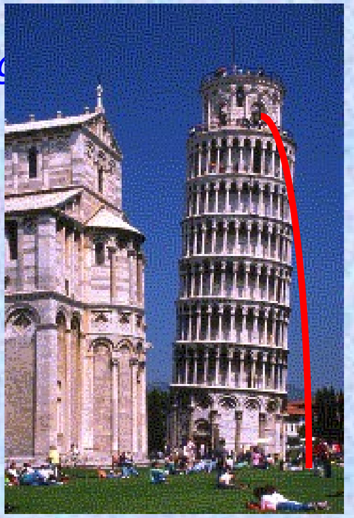
$$\boldsymbol{a} = -\frac{m_G}{m_I}\frac{GM\boldsymbol{r}}{r^3}$$

If $m_G = m_I(1+\eta)$

Then, comparing the acceleration of two different bodies or materials

$$\Delta \boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2 = -(\eta_1 - \eta_2)\frac{GM\boldsymbol{r}}{r^3}$$

The Weak Equivalence Principle (WEP)



400 CE Ioannes Philiponus: *"…let fall from the same heig*
two weights of which one is many times as heavy as the other …. the difference in time is a very small one"

1553 Giambattista Benedetti
proposed equality

1586 Simon Stevin
experiments

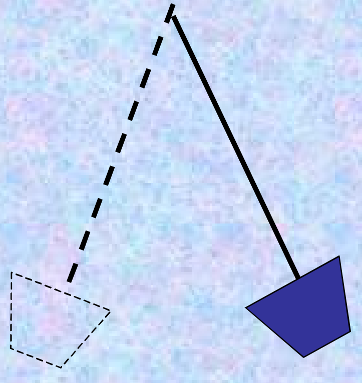
1589-92 Galileo Galilei
Leaning Tower of Pisa?

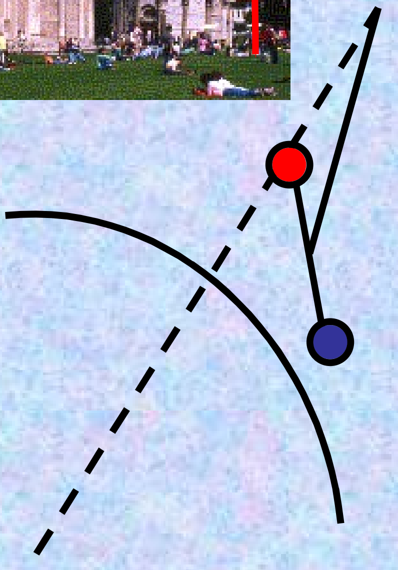
1670-87 Newton
pendulum experiments

1889, 1908 Baron R. von Eötvös
torsion balance experiments (10^{-9})

1990 – 2010 UW (Eöt-Wash)
10-13

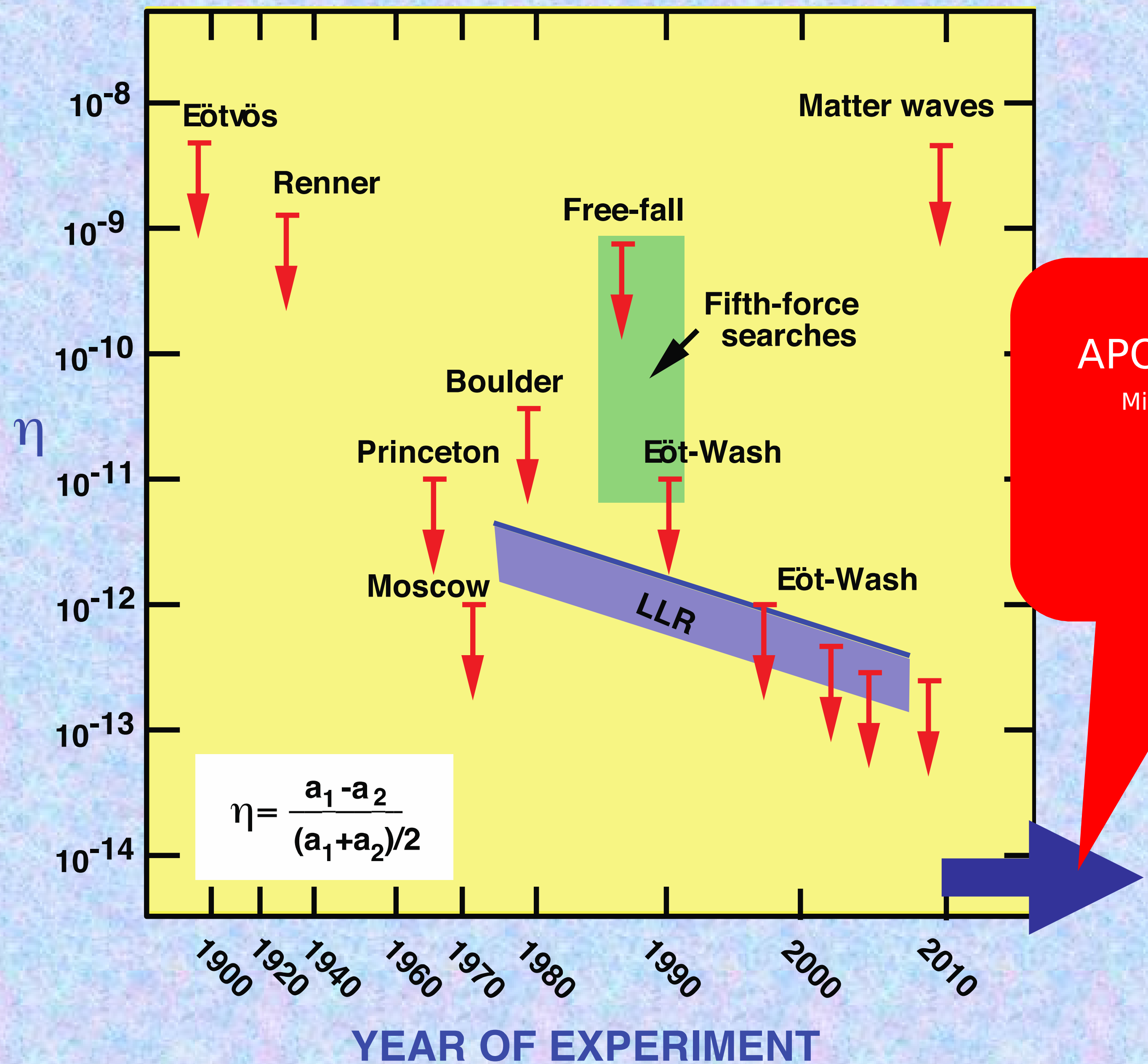
2010 Atom inteferometers
matter waves vs macroscopic object





Bodies fall in a gravitational field with an acceleration that is independent of mass, composition or internal structure

Tests of the Weak Equivalence Principle



APOLLO (LLR) 10^{-13}

Microscope 10-15 (2015)

Future: STEP, GG,

STE-QUEST

Newtonian equations of Hydrodynamics

Writing $m\boldsymbol{a} = m\boldsymbol{\nabla} U \,,$ Equation of motion

$U = GM/r \,,$ Field equation

Generalize to multiple sources (sum over M's) and continuous matt

$$\rho \frac{d\boldsymbol{v}}{dt} = \rho \boldsymbol{\nabla} U - \boldsymbol{\nabla} p \,,$$ Euler equation of motion

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \,,$$ Continuity equation

$$\nabla^2 U = -4\pi G \rho \,,$$ Poisson field equation

$$\frac{d}{dt} := \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \,,$$ Total or Lagrangian derivative

$$p = p(\rho, T, \dots)$$ Equation of state

Formal solution of Poisson's field equation:

Write $U(t, \boldsymbol{x}) = G \int G(\boldsymbol{x}, \boldsymbol{x}') \rho(t, \boldsymbol{x}') d^3x' \,,$

Green function $\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}') = -4\pi \delta(\boldsymbol{x} - \boldsymbol{x}') \Rightarrow G(\boldsymbol{x}, \boldsymbol{x}') = 1/|\boldsymbol{x} - \boldsymbol{x}'|$

$$U(t, \boldsymbol{x}) = G \int \frac{\rho(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, d^3x'$$

Rules of the road

Consequences of the continuity equation: for any f(x,t):

$$\frac{d}{dt}\int \rho(t,\boldsymbol{x})f(t,\boldsymbol{x})\,d^3x = \int\left(\rho\frac{\partial f}{\partial t} + f\frac{\partial \rho}{\partial t}\right)\,d^3x$$
$$= \int\left(\rho\frac{\partial f}{\partial t} - f\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\right)\,d^3x$$
$$= \int\left(\rho\frac{\partial f}{\partial t} + \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}f\right)\,d^3x - \oint f\rho\boldsymbol{v}\cdot d\boldsymbol{S}$$
$$= \int \rho\frac{df}{dt}\,d^3x\,.$$

Useful rules:

$$\frac{\partial}{\partial t}\int \rho(t,\boldsymbol{x'})f(t,\boldsymbol{x},\boldsymbol{x'})\,d^3x' = \int\rho'\left(\frac{\partial f}{\partial t} + \boldsymbol{v'}\cdot\boldsymbol{\nabla'}f\right)d^3x'\,,$$

$$\frac{d}{dt}\int \rho(t,\boldsymbol{x'})f(t,\boldsymbol{x},\boldsymbol{x'})\,d^3x' = \int\rho'\left(\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}f + \boldsymbol{v'}\cdot\boldsymbol{\nabla'}f\right)d^3x'$$
$$= \int\rho'\frac{df}{dt}\,d^3x'$$

Global conservation laws

$$M := \int \rho(t,\boldsymbol{x})\, d^3x = \text{constant}$$

$$\boldsymbol{P} := \int \rho(t,\boldsymbol{x})\boldsymbol{v}\, d^3x = \text{constant}$$

$$E := \mathcal{T}(t) + \Omega(t) + E_{\text{int}}(t) = \text{constant}$$

$$\boldsymbol{J} := \int \rho\boldsymbol{x}\times\boldsymbol{v}\, d^3x = \text{constant}$$

$$\boldsymbol{R}(t) := \frac{1}{M}\int \rho(t,\boldsymbol{x})\boldsymbol{x}\, d^3x = \frac{\boldsymbol{P}}{M}(t-t_0) + \boldsymbol{R}_0$$

$$d(\epsilon\mathcal{V}) + p\,d\mathcal{V} = 0$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \mathcal{V}^{-1}d\mathcal{V}/dt$$

$$\mathcal{T}(t) := \frac{1}{2}\int \rho v^2\, d^3x$$

$$\Omega(t) := -\frac{1}{2}G\int \frac{\rho\rho'}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' d^3x\,,$$

$$E_{\text{int}}(t) := \int \epsilon\, d^3x$$

$$\begin{aligned}\frac{d}{dt}\int \rho\boldsymbol{v}d^3x &= \int (\rho\boldsymbol{\nabla}U - \boldsymbol{\nabla}p)\, d^3x \\ &= -G\int\int \rho\rho'\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}d^3x d^3x' - \oint p\boldsymbol{n}d^2S \\ &= 0\end{aligned}$$

Spherical and nearly spherical bodies

Spherical symmetry

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) = -4\pi G\rho(t,r)$$

$$\frac{\partial U}{\partial r} = -\frac{Gm(t,r)}{r^2} \qquad m(t,r) := \int_0^r 4\pi\rho(t,r')r'^2\,dr'$$

$$U(t,r) = \frac{Gm(t,r)}{r} + 4\pi G\int_r^R \rho(t,r')r'\,dr'\,.$$

Outside the body $U = GM/r$

Spherical and nearly spherical bodies

Non-spherical bodies: the external field $|\boldsymbol{x}'| < |\boldsymbol{x}|$

Taylor expansion:

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|} = \frac{1}{r} - x'^j \partial_j \left(\frac{1}{r}\right) + \frac{1}{2} x'^j x'^k \partial_j \partial_k \left(\frac{1}{r}\right) - \cdots$$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} x'^L \partial_L \left(\frac{1}{r}\right)$$

Then the Newtonian potential outside the body becomes

$$U_{\rm ext}(t, \boldsymbol{x}) = G \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} I^{\langle L \rangle} \partial_{\langle L \rangle} \left(\frac{1}{r}\right) ,$$

$$I^{\langle L \rangle}(t) := \int \rho(t, \boldsymbol{x'}) x'^{\langle L \rangle} \, d^3 x'$$

$$x^L := x^i x^j \ldots (\text{L times})$$

$$\partial_L := \partial_i \partial_j \ldots (\text{L times})$$

$$\langle \ldots \rangle := \text{symmetric tracefree product}$$

Symmetric tracefree (STF) tensors

$A^{\langle ijk\ldots\rangle}$ Symmetric on all indices, and $\delta_{ij} A^{\langle ijk\ldots\rangle} = 0$

Example: gradients of 1/r

$$\partial_j r^{-1} = -n_j r^{-2}\,,$$

$$\partial_{jk} r^{-1} = \big(3 n_j n_k - \delta_{jk}\big) r^{-3}\,,$$

$$\partial_{jkn} r^{-1} = -\Big[15 n_j n_k n_n - 3\big(n_j \delta_{kn} + n_k \delta_{jn} + n_n \delta_{jk}\big)\Big] r^{-4}$$

$$\partial_L r^{-1} = \partial_{\langle L\rangle} r^{-1} = (-1)^\ell (2\ell-1)!!\,\frac{n_{\langle L\rangle}}{r^{\ell+1}}$$

General formula for $n^{<L>}$:

$$n^{\langle L\rangle} = \sum_{p=0}^{[\ell/2]} (-1)^p \frac{(2\ell-2p-1)!!}{(2\ell-1)!!}\Big[\delta^{2P} n^{L-2P} + \mathrm{sym}(q)\Big]$$

$$q := \ell!/[(\ell-2p)!(2p)!!]$$

Symmetric tracefree (STF) tensors

Link between $n^{<L>}$ and spherical harmonics

$$e_{\langle L\rangle} n^{\langle L\rangle} = \frac{\ell!}{(2\ell-1)!!} P_\ell(\boldsymbol{e}\cdot\boldsymbol{n})$$

$$n^{\langle L\rangle} := \frac{4\pi\ell!}{(2\ell+1)!!} \sum_{m=-\ell}^{\ell} \mathcal{Y}^{\langle L\rangle}_{\ell m} Y_{\ell m}(\theta,\phi)$$

$$\mathcal{Y}^{\langle z\rangle}_{10} = \sqrt{\frac{3}{4\pi}}, \qquad \mathcal{Y}^{\langle x\rangle}_{11} = -\sqrt{\frac{3}{8\pi}}, \qquad \mathcal{Y}^{\langle y\rangle}_{11} = i\sqrt{\frac{3}{8\pi}},$$

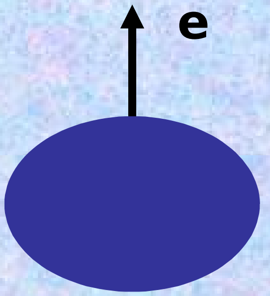
$$\mathcal{Y}^{\langle xx\rangle}_{20} = -\sqrt{\frac{5}{16\pi}}, \qquad \mathcal{Y}^{\langle yy\rangle}_{20} = -\sqrt{\frac{5}{16\pi}}, \qquad \mathcal{Y}^{\langle zz\rangle}_{20} = 2\sqrt{\frac{5}{16\pi}},$$

Average of $n^{<L>}$ over a sphere:

$$\langle\langle n^L\rangle\rangle := \frac{1}{4\pi}\oint n^L d\Omega = \begin{cases} \frac{1}{(2\ell+1)!!}\left(\delta^{L/2} + \mathrm{sym}[(\ell-1)!!]\right) & \ell = \mathrm{even} \\ 0 & \ell = \mathrm{odd} \end{cases}$$

Spherical and nearly spherical bodies

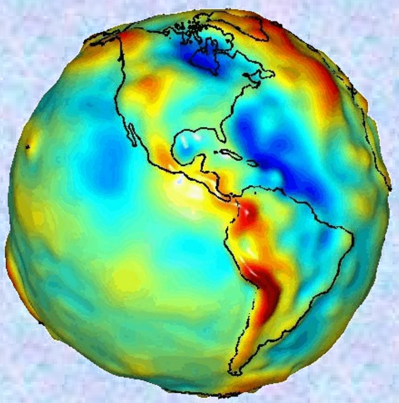
Example: axially symmetric body



$$I_A^{\langle L\rangle} = -m_A R_A^\ell (J_\ell)_A e^{\langle L\rangle}$$

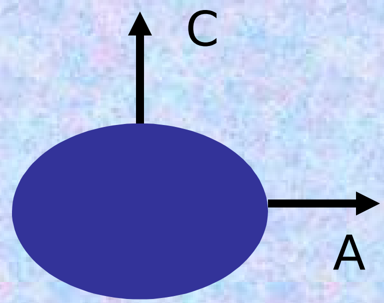
$$J_\ell := -\sqrt{\frac{4\pi}{2\ell+1}}\frac{1}{MR^\ell}\int \rho(t,\boldsymbol{x}) r^\ell Y^*_{\ell 0}(\theta,\phi)\, d^3x$$

$$U_{\rm ext}(t,\boldsymbol{x}) = \frac{GM}{r}\left[1 - \sum_{\ell=2}^{\infty} J_\ell \left(\frac{R}{r}\right)^\ell P_\ell(\cos\theta)\right]$$



Note that:

$$J_2 = \frac{C-A}{MR^2}$$



Motion of extended fluid bodies

Main assumptions:

- Bodies small compared to typical separation (R << r)
- "isolated" -- no mass flow
- $T_{int} \sim (R^3/Gm)^{1/2} << T_{orb} \sim (r^3/Gm)^{1/2}$ -- quasi equilibrium
- adiabatic response to tidal deformations -- nearly spherical

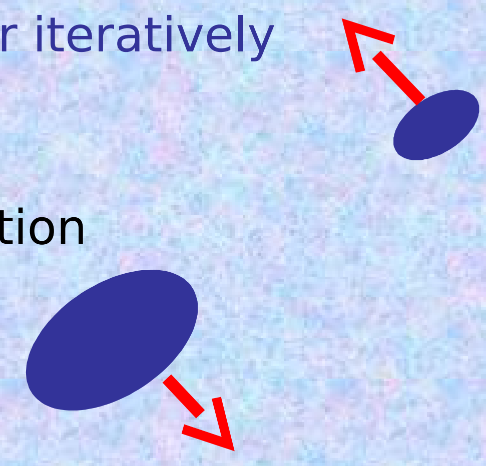
External problem:

- determine motions of bodies as functions (or functionals) of internal parameters

Internal problem:

- given motions, determine evolution of internal parameters

Solve the two problems self-consistently or iteratively



Example: Earth-Moon system -- orbital motion raises tides, tidally deformed fields affect motions

Motion of extended fluid bodies

Basic definitions

$$m_A := \int_A \rho(t,\boldsymbol{x})\, d^3x$$

$$\boldsymbol{r}_A(t) := \frac{1}{m_A}\int_A \rho(t,\boldsymbol{x})\boldsymbol{x}\, d^3x$$

$$dm_A/dt = 0$$

$$\boldsymbol{v}_A(t) := \frac{d\boldsymbol{r}_A}{dt} = \frac{1}{m_A}\int_A \rho\boldsymbol{v}\, d^3x$$

$$\boldsymbol{a}_A(t) := \frac{d\boldsymbol{v}_A}{dt} = \frac{1}{m_A}\int_A \rho\frac{d\boldsymbol{v}}{dt}\, d^3x$$

Is the center of mass unique?

- pure convenience, should not wander outside the body
- not physically measurable
- almost impossible to define in GR

$$\begin{aligned} m_A\boldsymbol{a}_A &= -G\int_A\int_A \rho\rho'\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}d^3xd^3x' \\ &\quad -G\int_A \rho\left[\sum_{B\neq A}\int_B \rho'\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}d^3x'\right]d^3x \end{aligned}$$

Define:

$$\boldsymbol{x} := \boldsymbol{r}_A(t) + \bar{\boldsymbol{x}}$$

$$\boldsymbol{x}' := \boldsymbol{r}_B(t) + \bar{\boldsymbol{x}}'$$

$$\boldsymbol{r}_{AB} := \boldsymbol{r}_A - \boldsymbol{r}_B$$

Motion of extended fluid bodies

N-body point mass terms

Moments of other bodies

Effect of body's own moments

Moment-moment interaction terms

$$a^j_A = G \sum_{B\neq A} \Bigg\{ -\frac{m_B}{r^2_{AB}} n^j_{AB}$$

$$+ \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \left[(-1)^\ell I_B^{\langle L\rangle} + \frac{m_B}{m_A} I_A^{\langle L\rangle} \right] \partial^A_{jL} \left(\frac{1}{r_{AB}} \right)$$

$$+ \frac{1}{m_A} \sum_{\ell=2}^{\infty} \sum_{\ell'=2}^{\infty} \frac{(-1)^{\ell'}}{\ell!\ell'!} I_A^{\langle L\rangle} I_B^{\langle L'\rangle} \partial^A_{jLL'} \left(\frac{1}{r_{AB}} \right) \Bigg\}$$

Two-body system with only body 2 having non-zero $I^{<L>}$

$$\boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2 \,, \quad r := |\boldsymbol{r}|$$

$$\boldsymbol{R} := (m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2)/m$$

$$m := m_1 + m_2$$

$$\mu := m_1 m_2 / m$$

$$a^j = -\frac{Gm}{r^2} n^j + Gm \sum_{\ell=2}^{\infty} \frac{(-1)^\ell}{\ell!} \frac{I_2^{\langle L\rangle}}{m_2} \partial_{jL} \left(\frac{1}{r} \right)$$

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$$
\begin{aligned}
a^j_A = G \sum_{B\neq A} \Bigg\{ & -\frac{m_B}{r^2_{AB}} n^j_{AB} \\
& + \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \left[(-1)^\ell I_B^{\langle L\rangle} + \frac{m_B}{m_A} I_A^{\langle L\rangle} \right] \partial^A_{jL} \left(\frac{1}{r_{AB}} \right) \\
& + \frac{1}{m_A} \sum_{\ell=2}^{\infty} \sum_{\ell'=2}^{\infty} \frac{(-1)^{\ell'}}{\ell!\ell'!} I_A^{\langle L\rangle} I_B^{\langle L'\rangle} \partial^A_{jLL'} \left(\frac{1}{r_{AB}} \right) \Bigg\}
\end{aligned}
$$

Two-body system with only body 2 having non-zero $I^{<L>}$

$$\boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2 \,, \quad r := |\boldsymbol{r}|$$

$$\boldsymbol{R} := (m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2)/m$$

$$m := m_1 + m_2$$

$$\mu := m_1 m_2 / m$$

$$a^j = -\frac{Gm}{r^2} n^j + Gm \sum_{\ell=2}^{\infty} \frac{(-1)^\ell}{\ell!} \frac{I_2^{\langle L\rangle}}{m_2} \partial_{jL} \left(\frac{1}{r} \right)$$

The two-body Kepler problem

- set center of mass at the origin (X = 0)
- ignore all multipole moments (spherical bodies or point masses)
- define $\boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2\,,\ r := |\boldsymbol{r}|\,,\ m := m_1 + m_2\,,\ \mu := m_1 m_2/m$
- reduces to effective one-body problem

$$\boldsymbol{a} = -\frac{Gm}{r^2}\boldsymbol{n}$$

Energy and angular momentum conserved:

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - G\frac{m_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}$$

$$= \frac{1}{2}\mu v^2 - G\frac{\mu m}{r}$$

$$L = m_1 \boldsymbol{r}_1 \times \boldsymbol{v}_1 + m_2 \boldsymbol{r}_2 \times \boldsymbol{v}_2$$

$$= \mu \boldsymbol{r} \times \boldsymbol{v}$$

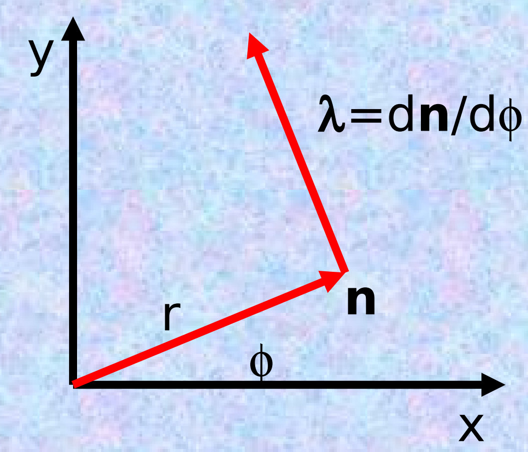
orbital plane
is fixed

Effective one-body problem

Make orbital plane the x-y plane

$$\boldsymbol{r} \times \boldsymbol{v} = r^2 \frac{d\phi}{dt} := h\boldsymbol{e}_z$$

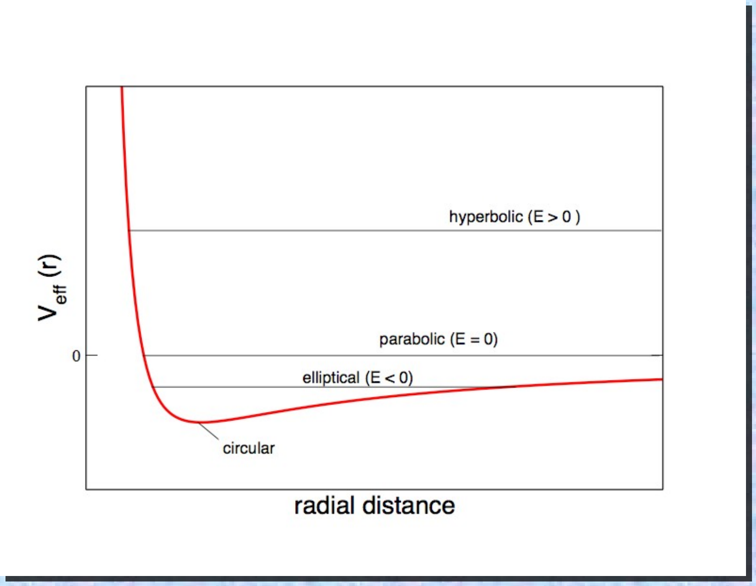
$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \dot{r}\boldsymbol{n} + r\dot{\phi}\boldsymbol{\lambda}$$



From energy conservation:

$$\dot{r}^2 = 2\left[\varepsilon - V_{\rm eff}(r)\right]$$

$$V_{\rm eff}(r) = \frac{h^2}{r^2} - \frac{Gm}{r}$$



Reduce to quadratures (integrals)

$$t - t_i = \pm \int_{r_i}^{r} \frac{dr'}{\sqrt{2[\varepsilon - V_{\rm eff}(r')]}}$$

$$\phi - \phi_i = h \int_{t_i}^{t} \frac{dt'}{r(t')^2}$$

Keplerian orbit solutions

Radial acceleration, or d/dt of energy equation:

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{Gm}{r^2}$$

Find the orbit in space: convert from t to φ:

$$d/dt = \dot{\phi} d/d\phi = (h/r^2) d/d\phi$$

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{Gm}{h^2}$$

$$\boxed{\frac{1}{r} = \frac{1}{p}(1 + e\cos f)}$$

$f := \phi - \omega$ true anomaly

$p := h^2/Gm$ semilatus rectum

Elliptical orbits (e < 1, a > 0)

$$r_{\rm peri} = \frac{p}{1+e}, \quad \phi = \omega$$

$$r_{\rm apo} = \frac{p}{1-e}, \quad \phi = \omega + \pi$$

$$a := \frac{1}{2}(r_{\rm peri} + r_{\rm apo}) = \frac{p}{1-e^2}$$

Hyperbolic orbits (e > 1, a < 0)

$$\phi_{\rm in} - \phi_{\rm out} = \pi - 2\arcsin(1/e)$$

Keplerian orbit solutions

Useful relationships

$$\dot{r} = \frac{he}{p} \sin f$$

$$v^2 = \frac{Gm}{p}(1 + 2e\cos f + e^2) = Gm\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$E = -\frac{G\mu m}{2a}$$

$$e^2 = 1 + \frac{2h^2 E}{\mu (Gm)^2}$$

$$P = 2\pi \left(\frac{a^3}{Gm}\right)^{1/2}$$ for closed orbits

Alternative solution

$$r = a(1 - e\cos u)$$

$$n(t - T) = u - e\sin u$$

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan\frac{u}{2}$$

$$n = 2\pi/P$$

u = eccentric anomaly

f = true anomaly

n = mean motion

Dynamical symmetry in the Kepler problem

- a and e are constant (related to E and h)
- orbital plane is constant (related to direction of h)
- ω is constant -- a hidden, dynamical symmetry

Runge-Lenz vector

$$\boldsymbol{A} := \frac{\boldsymbol{v} \times \boldsymbol{h}}{Gm} - \boldsymbol{n}$$

$$= e\left(\cos\omega\, \boldsymbol{e}_x + \sin\omega\, \boldsymbol{e}_y\right)$$

$$= \mathrm{constant}$$

Comments:

- responsible for the degeneracy of hydrogen energy levels
- added symmetry occurs only for 1/r and r^2 potentials
- deviation from 1/r potential generically causes dω/dt

Keplerian orbit in space

Six orbit elements:

- i = inclination relative to reference plane:

$$\cos\iota = \hat{\boldsymbol{h}} \cdot \boldsymbol{e}_Z$$

- Ω = angle of ascending node

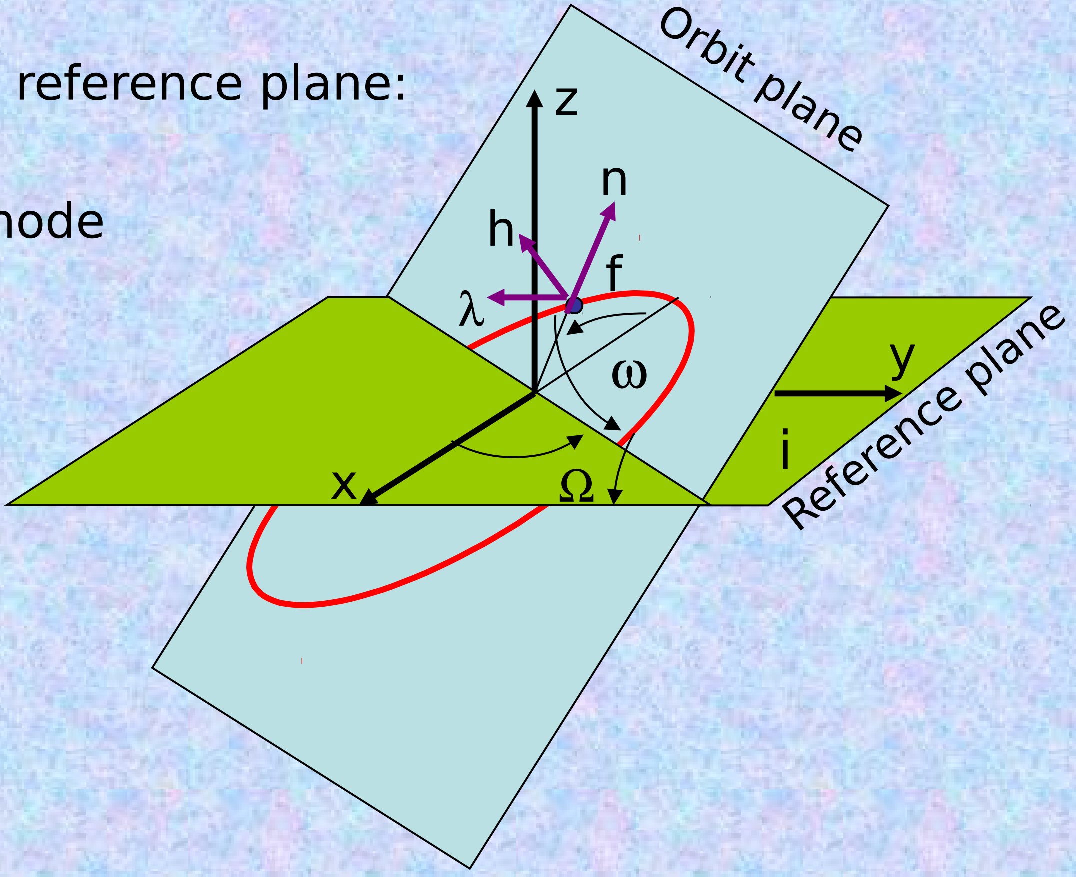
$$\cos\Omega = -\frac{\hat{\boldsymbol{h}} \cdot \boldsymbol{e}_Y}{\sin\iota}$$

- ω = angle of pericenter

$$\sin\omega = \frac{\boldsymbol{A} \cdot \boldsymbol{e}_z}{e\sin\iota}$$

- e = |**A**|
- a = h^2/Gm(1-e^{2})
- T = time of pericenter passage

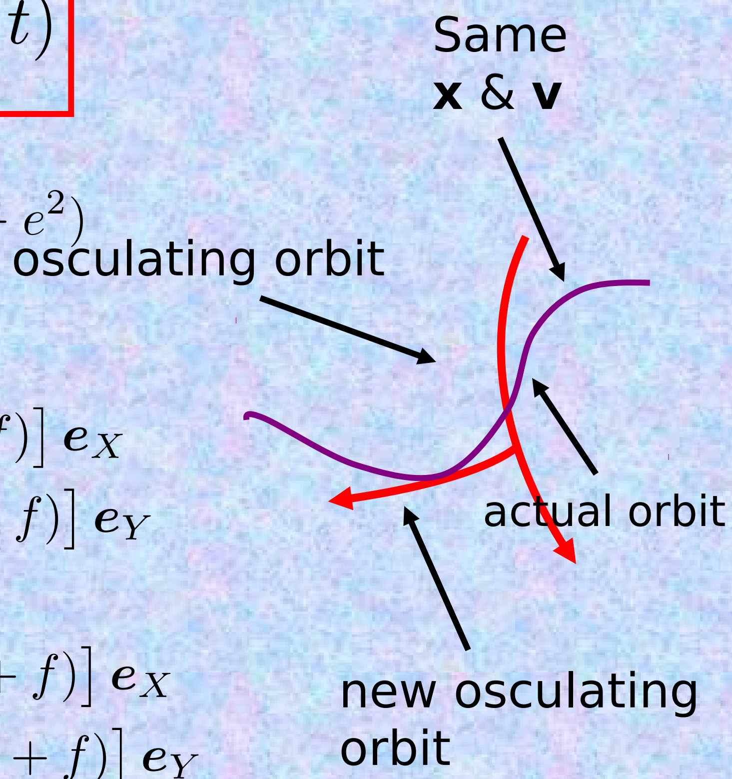
$$T = t - \int_0^f \frac{r^2}{h} df$$



Comment: equivalent to the initial conditions $\mathbf{x}_0$ and $\mathbf{v}0$

Osculating orbit elements and the perturbed Kepler problem

$$\boldsymbol{a} = -\frac{Gm\boldsymbol{r}}{r^3} + \boldsymbol{f}(\boldsymbol{r}, \boldsymbol{v}, t)$$



Define:

$$\boldsymbol{r} := r\boldsymbol{n}\,, \quad r := p/(1 + e\cos f)\,, \quad p = a(1 - e^2)$$

$$\boldsymbol{v} := \frac{he\sin f}{p}\boldsymbol{n} + \frac{h}{r}\boldsymbol{\lambda}\,, \quad h := \sqrt{Gmp}$$

$$\begin{aligned}
\boldsymbol{n} := &\left[\cos\Omega\cos(\omega + f) - \cos\iota\sin\Omega\sin(\omega + f)\right]\boldsymbol{e}_X \\
&+ \left[\sin\Omega\cos(\omega + f) + \cos\iota\cos\Omega\sin(\omega + f)\right]\boldsymbol{e}_Y \\
&+ \sin\iota\sin(\omega + f)\,\boldsymbol{e}_Z
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\lambda} := &\left[-\cos\Omega\sin(\omega + f) - \cos\iota\sin\Omega\cos(\omega + f)\right]\boldsymbol{e}_X \\
&+ \left[-\sin\Omega\sin(\omega + f) + \cos\iota\cos\Omega\cos(\omega + f)\right]\boldsymbol{e}_Y \\
&+ \sin\iota\cos(\omega + f)\,\boldsymbol{e}_Z
\end{aligned}$$

$$\hat{\boldsymbol{h}} := \boldsymbol{n} \times \boldsymbol{\lambda} = \sin\iota\sin\Omega\,\boldsymbol{e}_X - \sin\iota\cos\Omega\,\boldsymbol{e}_Y + \cos\iota\,\boldsymbol{e}_Z$$

e, a, ω, Ω, i, T may be functions of time

Perturbed Kepler problem

$$\boldsymbol{a} = -\frac{Gm\boldsymbol{r}}{r^3} + \boldsymbol{f}(\boldsymbol{r}, \boldsymbol{v}, t)$$

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v} \implies \frac{d\boldsymbol{h}}{dt} = \boldsymbol{r} \times \boldsymbol{f}$$

$$\boldsymbol{A} = \frac{\boldsymbol{v} \times \boldsymbol{h}}{Gm} - \boldsymbol{n} \implies Gm\frac{d\boldsymbol{A}}{dt} = \boldsymbol{f} \times \boldsymbol{h} + \boldsymbol{v} \times (\boldsymbol{r} \times \boldsymbol{f})$$

Decompose: $\boldsymbol{f} = \mathcal{R}\boldsymbol{n} + \mathcal{S}\boldsymbol{\lambda} + \mathcal{W}\hat{\boldsymbol{h}}$

$$\frac{d\boldsymbol{h}}{dt} = -r\mathcal{W}\,\boldsymbol{\lambda} + r\mathcal{S}\,\hat{\boldsymbol{h}}$$

$$Gm\frac{d\boldsymbol{A}}{dt} = 2h\mathcal{S}\,\boldsymbol{n} - \left(h\mathcal{R} + r\dot{r}\mathcal{S}\right)\boldsymbol{\lambda} - r\dot{r}\mathcal{W}\,\hat{\boldsymbol{h}}.$$

Example: $\dot{h} = r\mathcal{S}$

$$\frac{d}{dt}(h\cos\iota) = \dot{\boldsymbol{h}} \cdot \boldsymbol{e}_Z$$

$$\dot{h}\cos\iota - h\frac{d\iota}{dt}\sin\iota = -r\mathcal{W}\cos(\omega + f)\sin\iota + r\mathcal{S}\cos\iota$$

Perturbed Kepler problem

"Lagrange planetary equations"

$$\frac{dp}{dt} = 2\sqrt{\frac{p^3}{Gm}}\frac{1}{1+e\cos f}\,\mathcal{S},$$

$$\frac{de}{dt} = \sqrt{\frac{p}{Gm}}\left[\sin f\,\mathcal{R} + \frac{2\cos f + e(1+\cos^2 f)}{1+e\cos f}\,\mathcal{S}\right],$$

$$\frac{d\iota}{dt} = \sqrt{\frac{p}{Gm}}\frac{\cos(\omega+f)}{1+e\cos f}\,\mathcal{W},$$

$$\sin\iota\frac{d\Omega}{dt} = \sqrt{\frac{p}{Gm}}\frac{\sin(\omega+f)}{1+e\cos f}\,\mathcal{W},$$

$$\frac{d\omega}{dt} = \frac{1}{e}\sqrt{\frac{p}{Gm}}\left[-\cos f\,\mathcal{R} + \frac{2+e\cos f}{1+e\cos f}\sin f\,\mathcal{S} - e\cot\iota\frac{\sin(\omega+f)}{1+e\cos f}\,\mathcal{W}\right]$$

An alternative pericenter angle:

$$\varpi := \omega + \Omega\cos\iota$$

$$\frac{d\varpi}{dt} = \frac{1}{e}\sqrt{\frac{p}{Gm}}\left[-\cos f\,\mathcal{R} + \frac{2+e\cos f}{1+e\cos f}\sin f\,\mathcal{S}\right]$$

Perturbed Kepler problem

Comments:

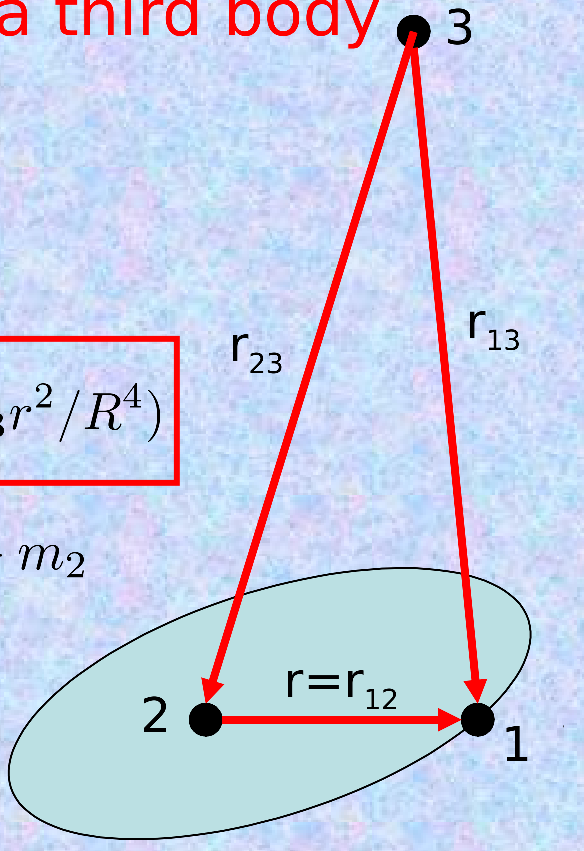
- these six 1st-order ODEs are exactly equivalent to the original three 2nd-order ODEs
- if **f** = 0, the orbit elements are constants
- if **f** << Gm/r2, use perturbation theory
- yields both periodic and secular changes in orbit elements
- can convert from d/dt to d/df using

$$\frac{df}{dt} = \left(\frac{df}{dt}\right)_{\text{Kepler}} - \left(\frac{d\omega}{dt} + \cos\iota\,\frac{d\Omega}{dt}\right)$$

Drop if working to 1st order

Perturbed Kepler problem

Worked example: perturbations by a third body



$$\boldsymbol{a}_1 = -Gm_2 \frac{\boldsymbol{r}_{12}}{r_{12}^3} - Gm_3 \frac{\boldsymbol{r}_{13}}{r_{13}^3},$$

$$\boldsymbol{a}_2 = +Gm_1 \frac{\boldsymbol{r}_{12}}{r_{12}^3} - Gm_3 \frac{\boldsymbol{r}_{23}}{r_{23}^3}$$

$$\boxed{\boldsymbol{a} = \frac{Gm\boldsymbol{r}}{r^3} - \frac{Gm_3 r}{R^3}\left[\boldsymbol{n} - 3(\boldsymbol{n}\cdot\boldsymbol{N})\boldsymbol{N}\right] + O(Gm_3 r^2/R^4)}$$

$$R := |\boldsymbol{r}_{23}|\,,\; N := \boldsymbol{r}_{23}/|\boldsymbol{r}_{23}|\,,\; m := m_1 + m_2$$

$$\mathcal{R} := \boldsymbol{f}\cdot\boldsymbol{n} = -\frac{Gm_3 r}{R^3}\left[1 - 3(\boldsymbol{n}\cdot\boldsymbol{N})^2\right],$$

$$\mathcal{S} := \boldsymbol{f}\cdot\boldsymbol{\lambda} = 3\frac{Gm_3 r}{R^3}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{\lambda}\cdot\boldsymbol{N}),$$

$$\mathcal{W} := \boldsymbol{f}\cdot\hat{\boldsymbol{h}} = 3\frac{Gm_3 r}{R^3}(\boldsymbol{n}\cdot\boldsymbol{N})(\hat{\boldsymbol{h}}\cdot\boldsymbol{N})$$

Put third body on a circular orbit

$$\boldsymbol{N} = \boldsymbol{e}_X \cos F + \boldsymbol{e}_Y \sin F\,, \qquad \frac{dF}{dt} = \sqrt{\frac{G(m+m_3)}{R^3}} \ll \frac{df}{dt}$$

Perturbed Kepler problem

Worked example: perturbations by a third body

Integrate over f from 0 to 2π holding F fixed, then average over F from 0 to 2π

$$\langle \Delta a \rangle = 0$$

$$\langle \Delta e \rangle = \frac{15\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 e(1-e^2)^{1/2}\sin^2\iota\sin\omega\cos\omega$$

$$\langle \Delta \omega \rangle = \frac{3\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 (1-e^2)^{-1/2}\left[5\cos^2\iota\sin^2\omega + (1-e^2)(5\cos^2\omega - 3)\right]$$

$$\langle \Delta \iota \rangle = -\frac{15\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 e^2(1-e^2)^{-1/2}\sin\iota\cos\iota\sin\omega\cos\omega$$

$$\langle \Delta \Omega \rangle = -\frac{3\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 (1-e^2)^{-1/2}(1-5e^2\cos^2\omega + 4e^2)\cos\iota$$

Also:

$$\langle \Delta \varpi \rangle = \frac{3\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 (1-e^2)^{1/2}\left[1+\sin^2\iota(1-5\sin^2\omega)\right]$$

Perturbed Kepler problem

Worked example: perturbations by a third body

Case 1: coplanar 3rd body and Mercury's perihelion (i = 0)

$$\langle \Delta\varpi \rangle = \frac{3\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 (1-e^2)^{1/2}$$

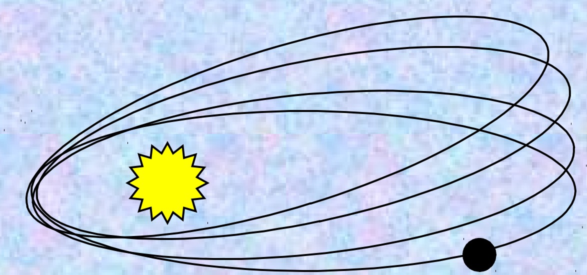
Planet	Semi-major axis (AU)	Orbital period (yr)	Eccentricity	Inclination to ecliptic $^\circ \cdot ' \cdot ''$	Inverse mass $1/M_\odot = 1$
Mercury	0.387099	0.24085	0.205628	7.0.15	6010000
Venus	0.723332	0.61521	0.006787	3.23.40	408400
Earth	1.000000	1.00004	0.016722	0.0.0	328910
Mars	1.523691	1.88089	0.093377	1.51.0	3098500
Jupiter	5.202803	11.86223	0.04845	1.18.17	1047.39
Saturn	9.53884	29.4577	0.05565	2.29.22	3498.5

For Jupiter:
$d\varpi/dt = 154 \,\text{as per century}\, (153.6)$

For Earth
$d\varpi/dt = 62 \,\text{as per century}\, (90)$

Mercury’s Perihelion: Trouble to Triumph

- **1687 Newtonian triumph**
- **1859 Leverrier’s conundrum**
- **1900 A turn-of-the century crisis**



575 “
per
century

Planet	Advance
Venus	277.8
Earth	90.0
Mars	2.5
Jupiter	153.6
Saturn	7.3
Total	531.2
Discrepancy	42.9
Modern measured value	42.98 ± 0.02
General relativity prediction	42.98

Perturbed Kepler problem

Worked example: perturbations by a third body

Case 2: the Kozai-Lidov mechanism

$$\langle \Delta a \rangle = 0$$

$$\langle \Delta e \rangle = \frac{15\pi}{2} \frac{m_3}{m} \left(\frac{a}{R} \right)^3 e(1-e^2)^{1/2} \sin^2 \iota \sin \omega \cos \omega$$

$$\langle \Delta \omega \rangle = \frac{3\pi}{2} \frac{m_3}{m} \left(\frac{a}{R} \right)^3 (1-e^2)^{-1/2} \left[5 \cos^2 \iota \sin^2 \omega + (1-e^2)(5 \cos^2 \omega - 3) \right]$$

$$\langle \Delta \iota \rangle = -\frac{15\pi}{2} \frac{m_3}{m} \left(\frac{a}{R} \right)^3 e^2 (1-e^2)^{-1/2} \sin \iota \cos \iota \sin \omega \cos \omega$$

Stationary point:

$$\omega_c = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$1 - e_c^2 = \frac{3}{5} \cos^2 \iota_c$$

A conserved quantity:

$$\frac{e}{1-e^2} \cos \iota \langle \Delta e \rangle + \sin \iota \langle \Delta \iota \rangle = 0$$

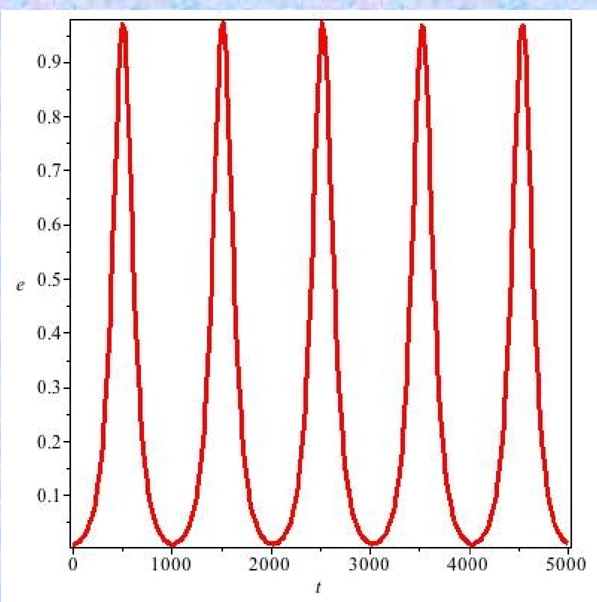
$$\Longrightarrow \sqrt{1-e^2} \cos \iota = \text{ constant}$$

L_Z !

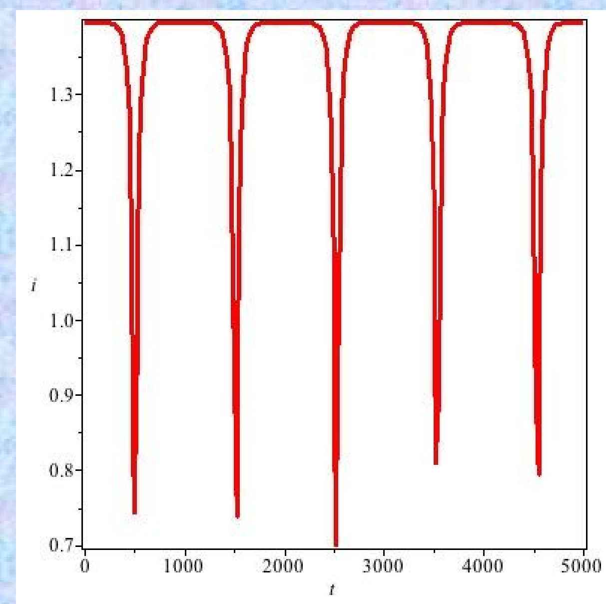
Perturbed Kepler problem

Worked example: perturbations by a third body

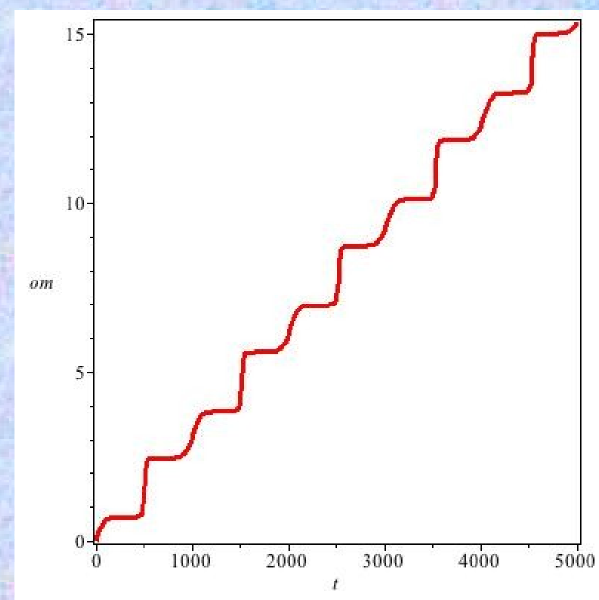
Case 2: the Kozai-Lidov mechanism



Eccentricity

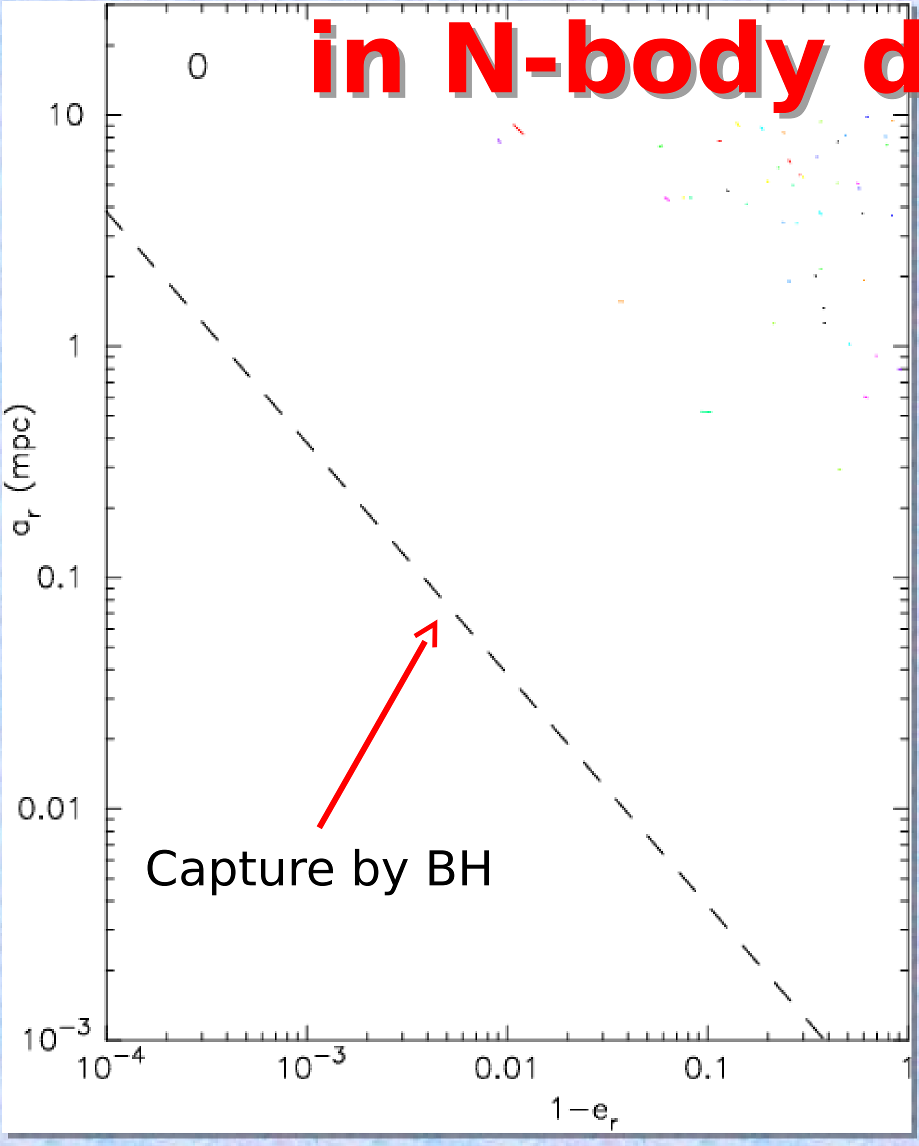


Inclination



Pericenter

Incorporating post-Newtonian effects in N-body dynamics



Merritt, Alexander, Mikkola & Will, PRD **84**, 044024 (2011)

Animations courtesy David Merritt

Perturbed Kepler problem

Worked example: body with a quadrupole moment

$$\boldsymbol{a} = \frac{Gm\boldsymbol{r}}{r^3} - \frac{3}{2}J_2\frac{GmR^2}{r^4}\Big\{\big[5(\boldsymbol{e}\cdot\boldsymbol{n})^2 - 1\big]\boldsymbol{n} - 2(\boldsymbol{e}\cdot\boldsymbol{n})\boldsymbol{e}\Big\},$$

$$\Delta a = 0,\ \Delta e = 0,\ \Delta\iota = 0$$

$$\Delta\omega = 6\pi J_2\left(\frac{R}{p}\right)^2\left(1 - \frac{5}{4}\sin^2\iota\right)$$

$$\Delta\Omega = -3\pi J_2\left(\frac{R}{p}\right)^2\cos\iota$$

For Mercury ($J_2 = 2.2 \times 10^{-7}$)

$$\frac{d\varpi}{dt} = 0.03\,\mathrm{as/century}$$

For Earth satellites ($J_2 = 1.08 \times 10^{-3}$)

$$\frac{d\Omega}{dt} = -3639\cos\iota\left(\frac{R}{a}\right)^{7/2}\ \mathrm{deg/yr}$$

- LAGEOS (a=1.93 R, i = 109^{o}.8): 120 deg/yr !
- Sun synchronous: a= 1.5 R, i = 65.9

Outline of the Lectures*

Part 1: Newtonian Gravity

- Foundations
- Equations of hydrodynamics
- Spherical and nearly spherical bodies
- Motion of extended fluid bodies

Part 2: Newtonian Celestial Mechanics

- Two-body Kepler problem
- Perturbed Kepler problem

*Based on *Gravity: Newtonian, post-Newtonian, General Relativistic*, by Eric Poisson and Clifford Will (Cambridge U Press, 2014)

FLORIDA GRAVITY

Outline of the Lectures*

Part 3: General Relativity

- Einstein equivalence principle
- GR field equations

Part 4: Post-Newtonian & post-Minkowskian theor

- Formulation
- Near-zone physics
- Wave-zone physics
- Radiation reaction

*Based on *Gravity: Newtonian, post-Newtonian, General Relativistic,* by Eric Poisson and Clifford Will (Cambridge U Press, 2014)

The Einstein Equivalence Principle

- Test bodies fall with the same acceleration
  Weak Equivalence Principle (WEP)
  - In a local freely falling frame, physics (nongravitational) is independent of frame’s velocity
  Local Lorentz Invariance (LLI)
  - In a local freely falling frame, physics (non-gravitational) is independent of frame’s location
  Local Position Invariance (LPI)

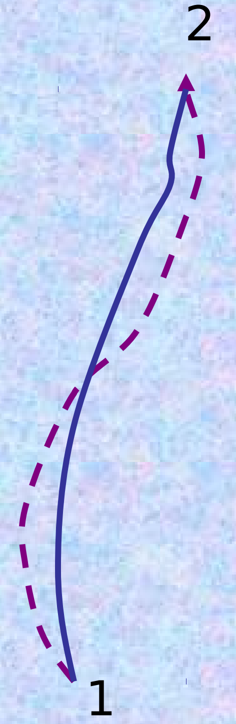
EEP => Metric theory of gravity

- $\eta_{\mu\nu}$ locally -> symmetric $g_{\mu\nu}$
- “comma” -> “semicolon”

Gravity = Geometry

“Curved spacetime tells matter how to move”

$$S = -mc^2 \int_1^2 d\tau$$
$$= -mc \int_1^2 \sqrt{-g_{\alpha\beta} \frac{dr^\alpha}{dt} \frac{dr^\beta}{dt}}\, dt$$



Euler-Lagrange equations (using τ as parameter):

$$\frac{d^2 r^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dr^\alpha}{d\tau} \frac{dr^\beta}{d\tau} = 0$$

Christoffel symbols

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \right).$$

“Gradient” of a vector

$$\nabla_\beta \vec{A} = (\partial_\beta A^\alpha) \vec{e}_\alpha + A^\alpha (\partial_\beta \vec{e}_\alpha)$$
$$= (\partial_\beta A^\alpha) \vec{e}_\alpha + A^\gamma \Gamma^\alpha{}_{\gamma\beta} \vec{e}_\alpha$$
$$= \nabla_\beta A^\alpha \vec{e}_\alpha$$

$$\vec{e}_\alpha \cdot \vec{e}_\beta = g_{\alpha\beta}$$

A geodesic parallel transports its own tangent vector $\nabla_{\vec{u}} \vec{u} = 0$

“Curved spacetime tells matter how to move”

Continuous matter, stress energy tensor

Perfect fluid: $T^{\alpha\beta} = (\rho c^2 + \epsilon + p) u^\alpha u^\beta / c^2 + p g^{\alpha\beta}$

$$j^\alpha = \rho u^\alpha$$

$$\boxed{\nabla_\beta T^{\alpha\beta} = 0\,,\ \nabla_\alpha j^\alpha = 0}$$

- ρ = rest mass density
- ε = energy density
- p = pressure
- u^α = four velocity

1st law of Thermodynamics

$$u_\alpha \nabla_\beta T^{\alpha\beta} = 0 = \frac{d\varepsilon}{d\tau} + (\varepsilon + p)\nabla \cdot \vec{u}$$

$$\boxed{d(\varepsilon \mathcal{V}) + p d\mathcal{V} = 0}$$

Relativistic Euler equation

$$(\mu + p)\frac{Du^\alpha}{d\tau} = -c^2\left(g^{\alpha\beta} + u^\alpha u^\beta / c^2\right)\nabla_\beta p$$

Compare with Newton

$$\rho \frac{d\boldsymbol{v}}{dt} + \boldsymbol{\nabla} U = -\boldsymbol{\nabla} p$$

"Matter tells spacetime how to curve"

Riemann tensor $R^{\alpha}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta}\Gamma^{\alpha}_{\beta\gamma}$

Ricci tensor $R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$

Ricci scalar $R = g^{\alpha\beta} R_{\alpha\beta}$

Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$

Bianchi identities $\nabla_\beta G^{\alpha\beta} = 0$

Action $S = \frac{c^3}{16\pi G} \int \sqrt{-g} R d^4x + S_{\mathrm{matter}}$

Einstein's equations: $G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}$

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Landau-Lifshitz Formulation of GR

Post-Newtonian and post-Minkowskian theory start with the Landau-Lifshitz formulation

Define the “gothic” metric density $\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta}$

Then Einstein’s equations can be written in the form

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = \frac{16\pi G}{c^4}(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\right)$$

$$H^{\alpha\mu\beta\nu} \equiv \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}$$

$$t^{\alpha\beta}_{\rm LL} \sim \partial\mathfrak{g} \cdot \partial\mathfrak{g}$$

Antisymmetry of $H^{\alpha\mu\beta\nu}$ implies the conservation equation

$$\partial_\beta \left[(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\right)\right] = 0 \quad \Longleftrightarrow \nabla_\beta T^{\alpha\beta} = 0$$

Landau-Lifshitz Formulation of GR

Conservation equation allows the formulation of global conservation laws:

$$E \equiv \int (-g)\big(T^{00} + t_{\rm LL}^{00}\big)\, d^3x$$

$$\frac{dE}{dt} = \oint (-g) t_{\rm LL}^{0j}\, d^2S_j$$

Similar conservation laws for linear momentum, angular momentum, and motion of a center of mass, with

$$P^j \equiv \frac{1}{c} \int (-g)\big(T^{j0} + t_{\rm LL}^{j0}\big)\, d^3x$$

$$J^j \equiv \frac{1}{c} \epsilon^{jkl} \int (-g) x^k \big(T^{l0} + t_{\rm LL}^{l0}\big)\, d^3x$$

$$X^j \equiv \frac{1}{E} \int (-g) x^j \big(T^{00} + t_{\rm LL}^{00}\big)\, d^3x$$

Landau-Lifshitz Formulation of GR

Define potentials $h^{\alpha\beta} \equiv \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}$

Impose a coordinate condition (gauge): Harmonic or deDonder gauge

$$\partial_\beta h^{\alpha\beta} = 0 \qquad \Box_g x^{(\alpha)} = 0$$

Matter tells spacetime how to curve

Spacetime tells matter how to move

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\tau^{\alpha\beta} \equiv (-g)\left(T^{\alpha\beta}[\mathsf{m}, g] + t^{\alpha\beta}_{\mathrm{LL}}[h] + t^{\alpha\beta}_{\mathrm{H}}[h]\right)$$

$$t^{\alpha\beta}_{\mathrm{H}} \sim \partial h \cdot \partial h + h\partial\partial h$$

$$\partial_\beta \tau^{\alpha\beta} = 0$$

Still equivalent to the exact Einstein equations

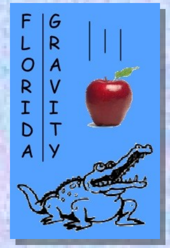
The "Relaxed" Einstein Equations

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

$$\partial_\beta \tau^{\alpha\beta} = 0$$

Solve for h as a functional of matter variables

Solve for evolution of matter variables to give h(t,x)



terating the “Relaxed” Einstein Equatio

Assume that $h^{\alpha\beta}$ is “small”, and iterate the relaxed equation:

$$\Box h^{\alpha\beta}_{N+1} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}(h_N)$$

$$h^{\alpha\beta}_{N+1} = \frac{4G}{c^4}\int \frac{\tau^{\alpha\beta}(h_N)(t-|\boldsymbol{x}-\boldsymbol{x}'|/c,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x'$$

Start with $h_0 = 0$ and truncate at a desired N

Yields an expansion in powers of G, called a post-Minkowskian expansion

Find the motion of matter using

$$\partial_\beta \tau^{\alpha\beta}(h_N) = 0$$

Solving the "Relaxed" Einstein Equations

$$\Box\psi = -4\pi\mu \Longrightarrow \psi = \int_{\mathcal{C}} \frac{\mu(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|} d^3x'$$

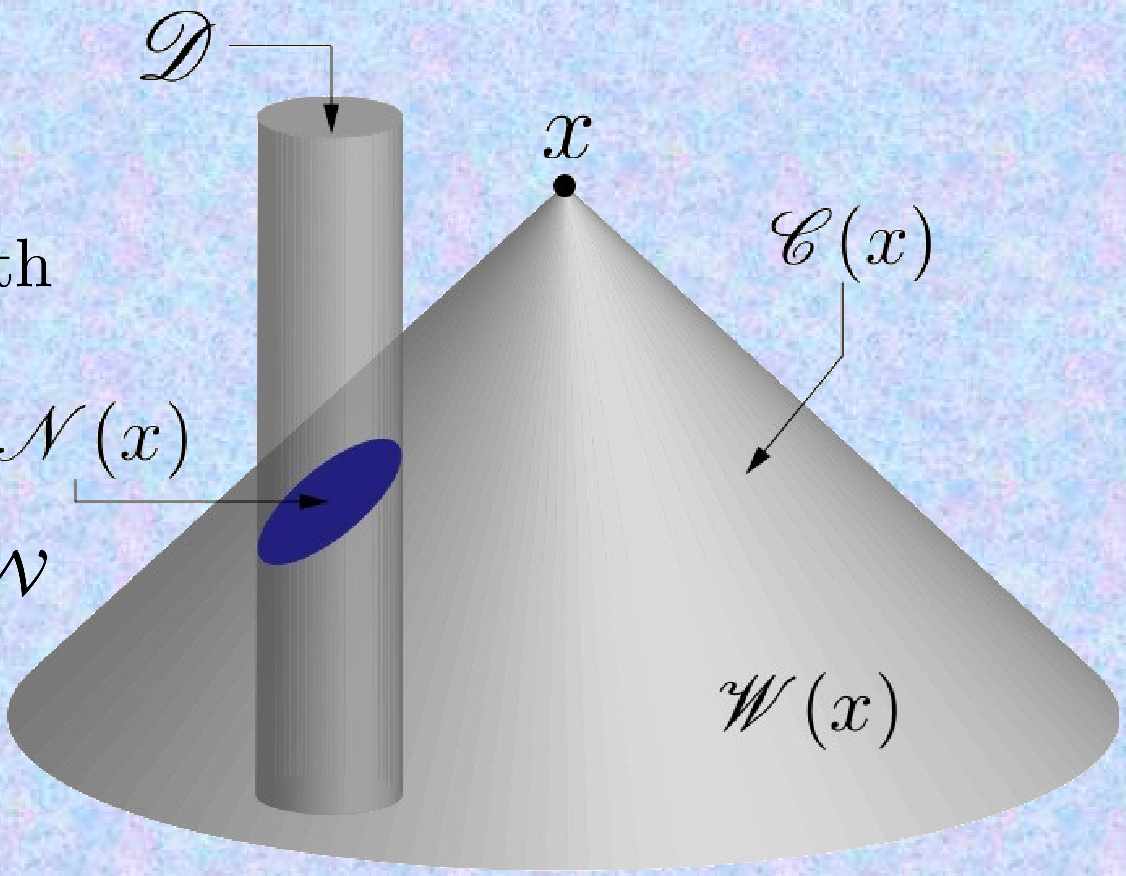
$\mathcal{N} : r' < \mathcal{R}\,,$

$\mathcal{W} : r' > \mathcal{R}$

$\mathcal{R} \sim$ wavelength

$\sim s/v$

$$\psi = \psi_{\mathcal{N}} + \psi_{\mathcal{W}}$$



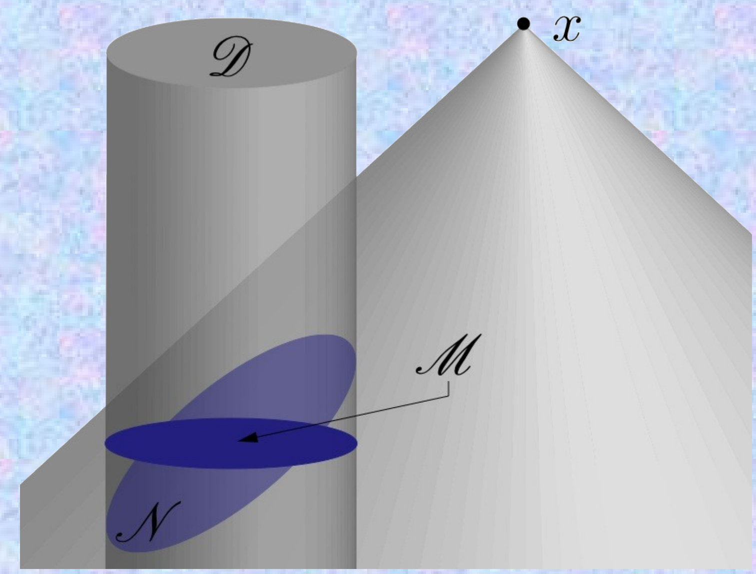
Solving the "Relaxed" Einstein Equations: Far zone

Near zone integral: $\psi_{\mathcal{N}}$

For x >> x', Taylor expand |x-x'|

$$\frac{\mu(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{x'}|}=\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}x'^L\partial_L\frac{\mu(t-r/c,\boldsymbol{y})}{r}.$$

$$\psi_{\mathcal{N}}(t,\boldsymbol{x})=\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\mu(\tau,\boldsymbol{x'})x'^L\,d^3x'\right]$$



A multipole expansion

$$\tau=t-R/c$$

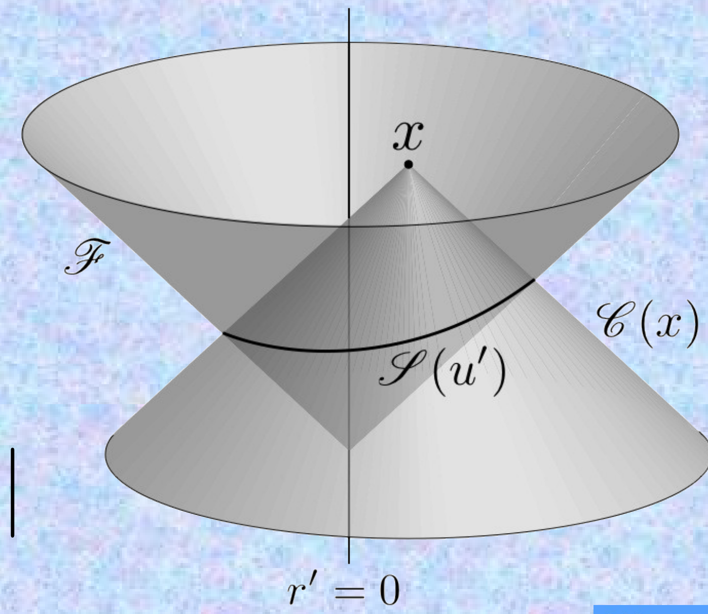
Integrals depend on R

Solving the “Relaxed” Einstein Equations
Far zone

Far zone integral: $\psi \mathcal{W}$

ince contributions to μ in the far zone come from retarded fields,
ave the generic form

$$\mu \sim f(\tau', \theta', \phi')/r'^n$$

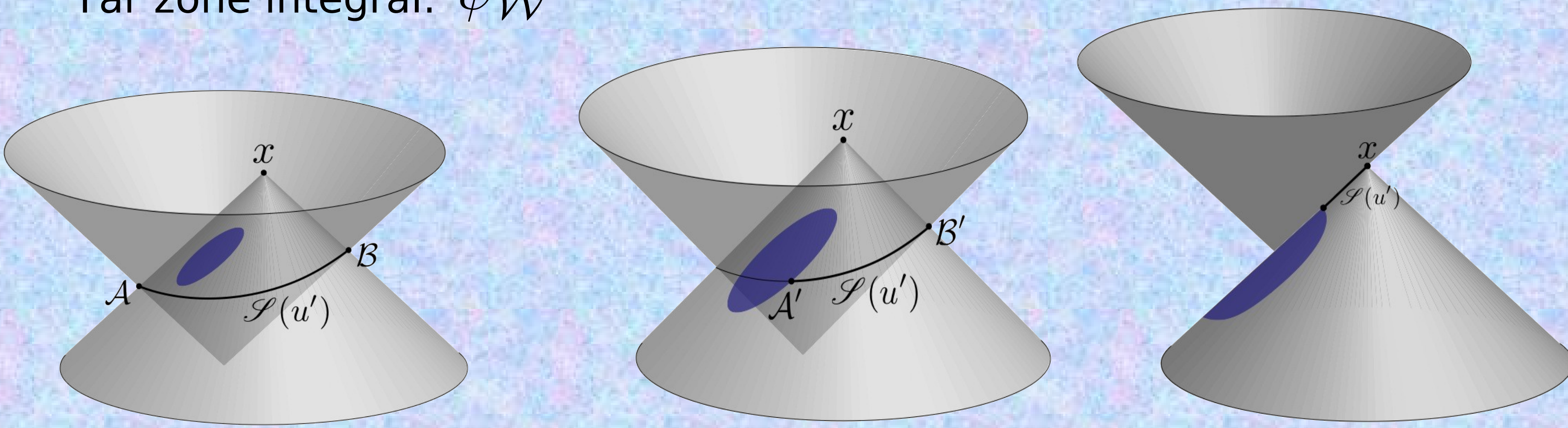


Change variables from (r’, θ’, ϕ’)
to (u’, θ’, ϕ’), where u’ = cτ’ = ct’-r’

$$u' + r' = ct - |\boldsymbol{x} - \boldsymbol{x}'|$$

Solving the "Relaxed" Einstein Equations
Far zone

Far zone integral: $\psi_{\mathcal{W}}$

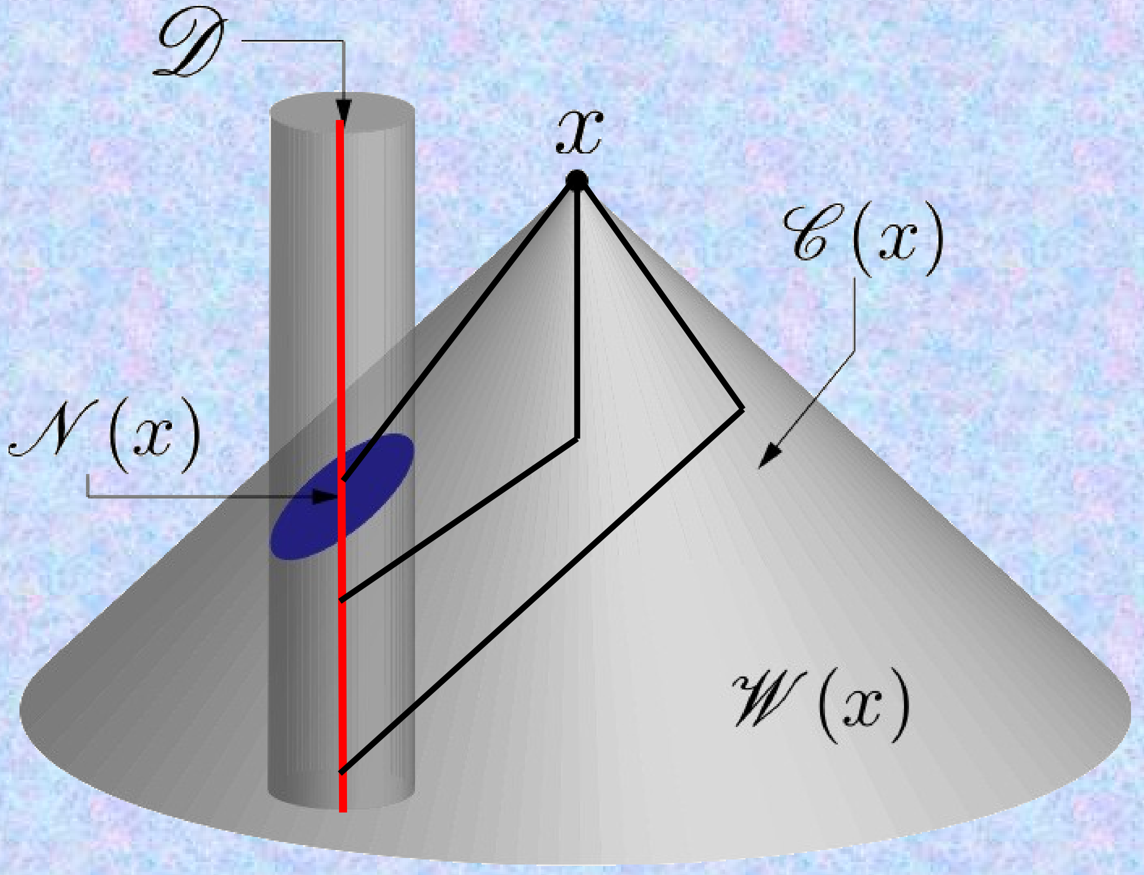


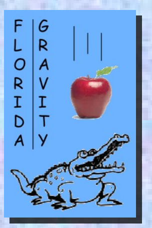
$$\psi_{\mathcal{W}} = \frac{1}{4\pi}\int_{-\infty}^{u} du' \oint_{\mathcal{S}(u')} \frac{f(u'/c,\theta',\phi')}{r'(u',\theta',\phi')^{n-2}} \frac{d\Omega'}{ct - u' - \boldsymbol{n'}\cdot\boldsymbol{x}}$$

Integral also depends on R

But $\psi = \psi_{\mathcal{N}} + \psi_{\mathcal{W}}$ is independent of R

Gravity as a source of gravity and gravitational "tails"





Solving the “Relaxed” Einstein Equations
Near zone

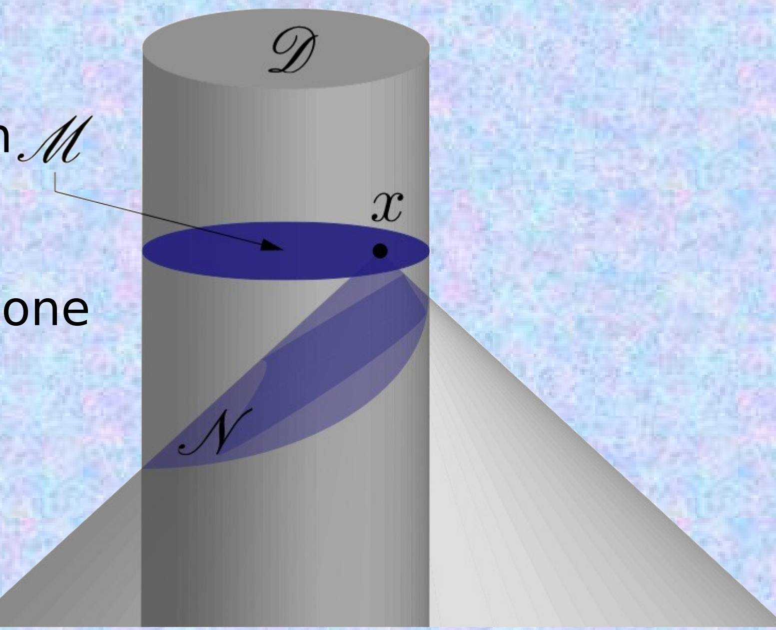
Near zone integral: $\psi_{\mathcal{N}}$

For x ~ x’, Taylor expand about t

$$\mu(t - |\boldsymbol{x} - \boldsymbol{x'}|/c) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell! c^\ell} \left(\frac{\partial}{\partial t} \right)^\ell \mu(t, \boldsymbol{x'}) |\boldsymbol{x} - \boldsymbol{x'}|^\ell$$

$$\psi_{\mathcal{N}}(t, \boldsymbol{x}) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell! c^\ell} \left(\frac{\partial}{\partial t} \right)^\ell \int_{\mathcal{M}} \mu(t, \boldsymbol{x'}) |\boldsymbol{x} - \boldsymbol{x'}|^{\ell - 1} \, d^3x'$$

- A post-Newtonian expansion in powers of 1/c
- Instantaneous potentials
- Must also calculate the far-zone integral $\psi_{\mathcal{W}}$



Outline of the Lectures*

Part 3: General Relativity

- Einstein equivalence principle
- GR field equations

Part 4: Post-Newtonian & post-Minkowskian theor

- Formulation
- Near-zone physics
- Wave-zone physics
- Radiation reaction

*Based on *Gravity: Newtonian, post-Newtonian, General Relativistic,* by Eric Poisson and Clifford Will (Cambridge U Press, 2014)

Near zone physics; Motion of extended fluid bodies

Matter variables:

$$\text{rescaled mass density}: \rho^* \equiv \rho\sqrt{-g}(u^0/c)$$
$$\text{proper pressure}: p$$
$$\text{internal energy per unit mass}: \Pi$$
$$\text{four} - \text{velocity of fluid element}: u^\alpha = u^0(1, \boldsymbol{v}/c)$$

$$\nabla_\alpha(\rho u^\alpha) = 0 \Longleftrightarrow \frac{\partial \rho^*}{\partial t} + \boldsymbol{\nabla}(\rho^* \boldsymbol{v}) = 0$$

Slow-motion assumption v/c << 1:

$$T^{0j}/T^{00} \sim v/c\,, \qquad T^{jk}/T^{00} \sim (v/c)^2$$
$$h^{0j}/h^{00} \sim v/c\,, \qquad h^{jk}/h^{00} \sim (v/c)^2$$

post-Newtonian approximation: Near zone

Recall the action for a geodesic

$$S = -mc^2 \int_1^2 d\tau$$

$$= -mc \int_1^2 \sqrt{-g_{\alpha\beta}\frac{dr^\alpha}{dt}\frac{dr^\beta}{dt}}\, dt$$

$$= -mc \int_1^2 \left(1 - 2\frac{U}{c^2} - \delta g_{00} - 2\frac{v^j}{c}\delta g_{0j} - \frac{v^2}{c^2} + \frac{v^i v^j}{c^2}\delta g_{ij}\right)^{1/2} dt$$

ϵ $\quad \epsilon^2$ $\quad \epsilon^2$ $\quad \epsilon$ $\quad \epsilon^2$

$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \sim \epsilon$$

We need to calculate

$$\delta g_{00} \quad \text{to} \quad O(\epsilon^2)$$

$$\delta g_{0j} \quad \text{to} \quad O(\epsilon^{3/2})$$

$$\delta g_{ij} \quad \text{to} \quad O(\epsilon)$$

Two iterations of the relaxed equations required

Post-Newtonian limit of general relativity

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\psi + \frac{1}{2}\partial_{tt}X - U^2\right) + O(c^{-6})\,,$$

$$g_{0j} = -\frac{4}{c^3}U_j + O(c^{-5})\,,$$

$$g_{jk} = \delta_{jk}\left(1 + \frac{2}{c^2}U\right) + O(c^{-4})\,,$$

b

g

$$U(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x'\,,$$

$$\psi(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}\left(\frac{3}{2}v'^2 - U' + \Pi' + 3p'/\rho^{*\prime}\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x'\,,$$

$$X(t,\boldsymbol{x}) := G\int \rho^{*\prime}|\boldsymbol{x}-\boldsymbol{x}'|\, d^3x'\,,$$

$$U^j(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}v'^j}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x'$$

Bounds on the PPN Parameters

Parameter	Effect or Experiment	Bound	Remarks
$\gamma - 1$	Time delay	2.3×10^{-5}	Cassini tracking
	Light deflection	2×10^{-4}	VLBI
$\beta - 1$	Perihelion shift	8×10^{-5}	$J_2 = 2.2 \times 10^{-7}$
	Nordtvedt effect	2.3×10^{-4}	LLR, $\eta < 3 \times 10^{-4}$
ξ	Spin Precession	4×10^{-9}	Millisecond pulsars
α_1	Orbit polarization	10^{-4}	LLR
		4×10^{-5}	Pulsar J 1738+0333
α_2	Spin precession	2×10^{-9}	Millisecond pulsars
α_3	Self-acceleration	4×10^{-20}	Pulsar spindown
ζ_1	--	2×10^{-2}	Combined bounds
ζ_2	Binary acceleration	4×10^{-5}	PSR 1913+16
ζ_3	Newton's 3rd law	10^{-8}	Lunar acceleration
ζ_4	--		Not independent

$$\eta = 4\beta - \gamma - 3 - 10\xi/3 - \alpha_1 + 2\alpha_2/3 - 2\zeta_1/3 - \zeta_2/3$$

Bound on scalar-tensor gravity: $\omega > 40{,}000$

Post-Newtonian Hydrodynamics

From $\nabla_\beta T^{\alpha\beta} = 0$

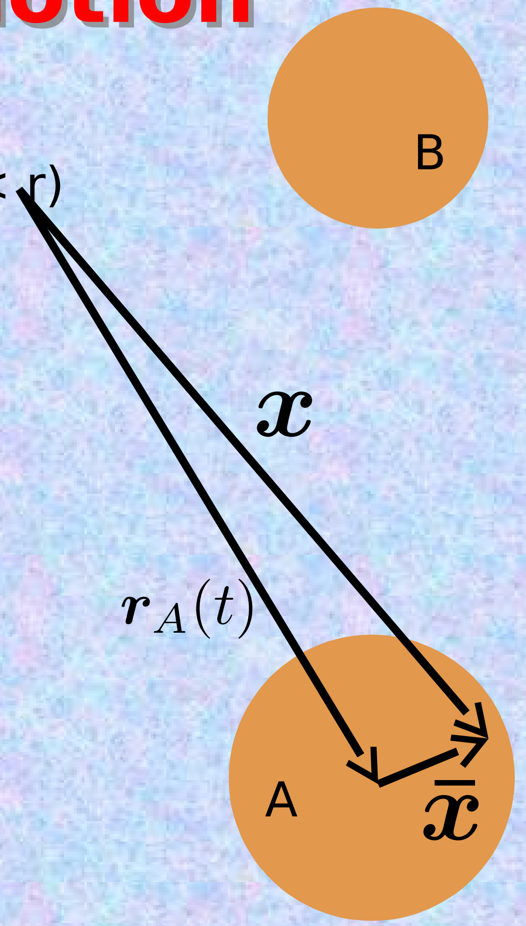
Post-Newtonian equation of hydrodynamics

$$\begin{aligned}
\rho^* \frac{dv^j}{dt} &= -\partial_j p + \rho^* \partial_j U \\
&\quad + \frac{1}{c^2}\left[\left(\frac{1}{2}v^2 + U + \Pi + \frac{p}{\rho^*}\right)\partial_j p - v^j \partial_t p\right] \\
&\quad + \frac{1}{c^2}\rho^*\Big[(v^2 - 4U)\partial_j U - v^j\big(3\partial_t U + 4v^k \partial_k U\big) \\
&\qquad + 4\partial_t U_j + 4v^k\big(\partial_k U_j - \partial_j U_k\big) + \partial_j \Psi\Big] \\
&\quad + O(c^{-4})
\end{aligned}$$

N-body equations of motion

Main assumptions:

- Bodies small compared to typical separation (R << r)
- "isolated" -- no mass flow
- ignore contributions that scale as R^n
- assume bodies are reflection symmetric



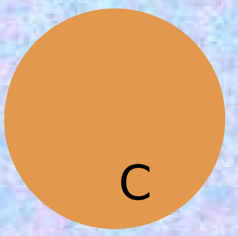
$$\text{mass}: \quad m_A \equiv \int_A \rho^* d^3x$$

$$\text{position}: \quad \boldsymbol{r}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{x} d^3x$$

$$\text{velocity}: \quad \boldsymbol{v}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{v} d^3x = \frac{d\boldsymbol{r}_A}{dt}$$

$$\text{acceleration}: \quad \boldsymbol{a}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{a} d^3x = \frac{d\boldsymbol{v}_A}{dt}$$

$$\boldsymbol{x} \equiv \boldsymbol{r}_A(t) + \bar{\boldsymbol{x}}$$



N-body equations of motion

Dependence on internal structure?

$$\mathcal{T}_A \equiv \frac{1}{2}\int_A \rho^* \bar{v}^2\, d^3\bar{x}, \qquad P_A \equiv \int_A p\, d^3\bar{x}\,,$$

$$\Omega_A \equiv -\frac{1}{2}G\int_A \frac{\rho^*\rho^{*\prime}}{|\bar{\boldsymbol{x}} - \bar{\boldsymbol{x}}'|}\, d^3\bar{x}'\, d^3\bar{x}, \qquad E_A^{\rm int} \equiv \int_A \rho^*\Pi\, d^3\bar{x}$$

Use the virial theorem:

$$2\mathcal{T}_A + \Omega_A + 3P_A = 0$$

Then all structure integrals can be absorbed into a single “total” mass:

$$M_A \equiv m_A + \frac{1}{c^2}\left(\mathcal{T}_A + \Omega_A + E_A^{\rm int}\right) + O(c^{-4})$$

This is a manifestation of the **Strong Equivalence Principle,** satisfied by GR, but not by most alternative theories.
The motions of **all** bodies, including NS and BH, are independent of their internal structure – in GR!

N-body equations of motion

$$
\begin{aligned}
\boldsymbol{a}_A = &- \sum_{B\neq A} \frac{GM_B}{r_{AB}^2}\boldsymbol{n}_{AB} \\
&+ \frac{1}{c^2}\Bigg(- \sum_{B\neq A} \frac{GM_B}{r_{AB}^2}\Bigg[v_A^2 - 4(\boldsymbol{v}_A\cdot\boldsymbol{v}_B) + 2v_B^2 - \frac{3}{2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B)^2 \\
&\qquad\qquad\qquad\qquad - \frac{5GM_A}{r_{AB}} - \frac{4GM_B}{r_{AB}}\Bigg]\boldsymbol{n}_{AB} \\
&+ \sum_{B\neq A} \frac{GM_B}{r_{AB}^2}\Big[\boldsymbol{n}_{AB}\cdot(4\boldsymbol{v}_A - 3\boldsymbol{v}_B)\Big](\boldsymbol{v}_A - \boldsymbol{v}_B) \\
&+ \sum_{B\neq A}\sum_{C\neq A,B} \frac{G^2M_BM_C}{r_{AB}^2}\Bigg[\frac{4}{r_{AC}} + \frac{1}{r_{BC}} - \frac{r_{AB}}{2r_{BC}^2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{n}_{BC})\Bigg]\boldsymbol{n}_{AB} \\
&- \frac{7}{2}\sum_{B\neq A}\sum_{C\neq A,B} \frac{G^2M_BM_C}{r_{AB}r_{BC}^2}\boldsymbol{n}_{BC}\Bigg) + O(c^{-4}).
\end{aligned}
$$

N-body equations of motion:
orked example: 2 bodies and the perihelion sh

Define:

$$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad m \equiv M_1 + M_2$$

$$\boldsymbol{v} \equiv \boldsymbol{v}_1 - \boldsymbol{v}_2 \qquad \eta \equiv \frac{M_1 M_2}{(M_1 + M_2)^2}$$

$$\boldsymbol{a} \equiv \boldsymbol{a}_1 - \boldsymbol{a}_2$$

$$\boldsymbol{n} \equiv \boldsymbol{r}/r$$

$$\dot{r} \equiv dr/dt = \boldsymbol{n} \cdot \boldsymbol{v}$$

$$\boldsymbol{a} = -\frac{Gm}{r^2}\boldsymbol{n} - \frac{Gm}{c^2 r^2}\left\{\left[(1+3\eta)v^2 - \frac{3}{2}\eta\dot{r}^2 - 2(2+\eta)\frac{Gm}{r}\right]\boldsymbol{n} - 2(2-\eta)\dot{r}\boldsymbol{v}\right\} + O(c^{-4}),$$

N-body equations of motion:
orked example: 2 bodies and the perihelion sh

Components of the disturbing force

$$\mathcal{R} = \frac{Gm}{c^2r^2}\left[-(1+3\eta)v^2 + \frac{1}{2}(8-\eta)\dot{r}^2 + 2(2+\eta)\frac{Gm}{r}\right],$$

$$\mathcal{S} = \frac{Gm}{c^2r^2}\left[2(2-\eta)\dot{r}(r\dot{\phi})\right],$$

$$\mathcal{W} = 0$$

Integrate the Lagrange planetary equations:

$$\Delta e = \Delta a = 0$$

$$\Delta\Omega = \Delta\iota = 0$$

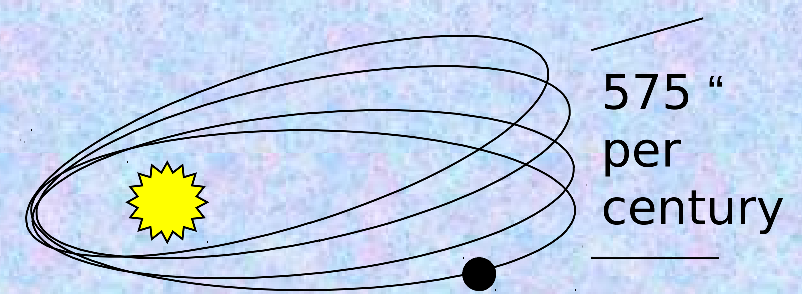
$$\Delta\omega = \frac{6\pi G(M_1+M_2)}{a(1-e^2)c^2}$$

42.98 “/c for Mercury

4.226598 O/yr for PSR 1913+16

Mercury's Perihelion: Trouble to Triumph

- **1687 Newtonian triumph**
- **1859 Leverrier's conundrum**
- **1900 A turn-of-the century crisis**



Planet	Advance
Venus	277.8
Earth	90.0
Mars	2.5
Jupiter	153.6
Saturn	7.3
Total	531.2
Discrepancy	42.9
Modern measured value	42.98 ± 0.02
General relativity prediction	42.98

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terating the “Relaxed” Einstein Equatio

Assume that $h^{\alpha\beta}$ is “small”, and iterate the relaxed equation:

$$\Box h^{\alpha\beta}_{N+1} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}(h_N)$$

$$h^{\alpha\beta}_{N+1} = \frac{4G}{c^4}\int \frac{\tau^{\alpha\beta}(h_N)(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|} d^3x'$$

Start with $h_0 = 0$ and truncate at a desired N

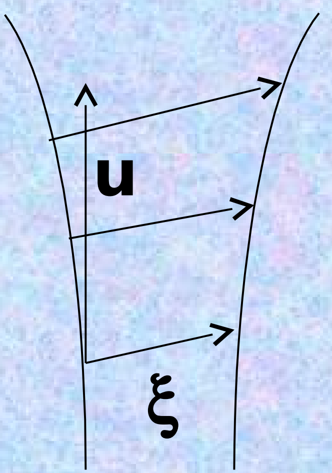
Yields an expansion in powers of G, called a post-Minkowskian expansion

Find the motion of matter using

$$\partial_\beta \tau^{\alpha\beta}(h_N) = 0$$

Wave Zone Physics: Gravitational Waves

Geodesic deviation: $\frac{D^2\xi^\alpha}{ds^2} = -R^\alpha{}_{\beta\gamma\delta} u^\beta \xi^\gamma u^\delta$

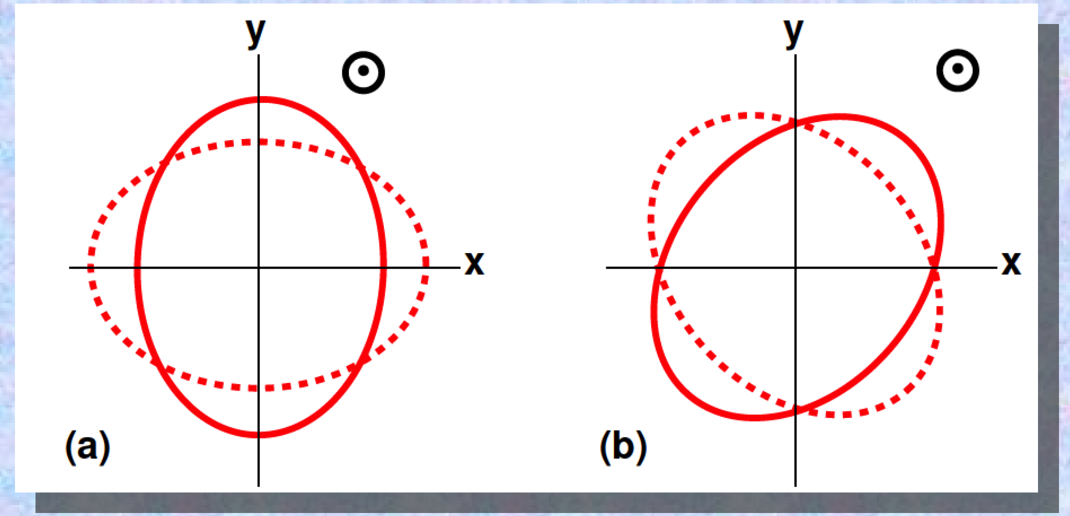


In the rest frame of an observer

$$\frac{d^2\xi^j}{dt^2} = -c^2 R_{0j0k}\xi^k$$
$$= \frac{1}{2}\partial_{\tau\tau} h^{jk}_{TT}\xi^k \qquad \tau = t - R/c$$

$$h^{jk}_{TT} \equiv \left(P^j{}_p P^k{}_q - \frac{1}{2} P^{jk} P_{pq} \right) h^{pq} , \qquad P^j_p = \delta^j_p - N^j N_p$$

$$N_j h^{ij}_{TT} = 0$$
$$\delta_{jk} h^{ij}_{TT} = 0$$



Wave Zone Physics: Gravitational Waves

The quadrupole formula:

Requires **two** iterations of the relaxed Einstein equation:

$$h_{2\ \mathrm{TT}}^{ij} = \frac{4G}{c^4} \int \frac{\tau^{ij}(h_1)(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{x'})_{\mathrm{TT}}}{|\boldsymbol{x} - \boldsymbol{x'}|} d^3x'$$

$$\rightarrow \frac{2G}{Rc^4} \ddot{I}^{\langle ij \rangle}_{\mathrm{TT}}(t - R/c) \text{ in the far wave-zone}$$

$$I^{\langle ij \rangle}(t) = \int \rho^*(t, \boldsymbol{x}) \left(x^i x^j - \frac{1}{3} r^2 \delta^{ij} \right) d^3x$$

For an N-body system $\ddot{I}^{\langle ij \rangle} = 2 \sum_A M_A v_A^{\langle ij \rangle} - \sum_{A \neq B} \frac{G M_A M_B}{r_{AB}} n_{AB}^{\langle ij \rangle}$

- By convention, the quadrupole formula is called the ``Newtonian''-order result
- Higher order PN corrections can be calculated by further iterating the relaxed equations
- 3 iterations needed for 1 & 1.5 PN order, 4 for 2PN order etc

Wave Zone Physics: Gravitational Waves

Beyond the quadrupole formula:

For a binary system in a circular orbit:

$$h_{+,\times} = \frac{2\eta G m}{c^2 R}\beta^2 \left[(1+2\pi\beta^3)H^{[0]}_{+,\times} + \Delta\beta H^{[1/2]}_{+,\times} + \beta^2 H^{[1]}_{+,\times} + \Delta\beta^3 H^{[3/2]}_{+,\times} + O(\beta^4)\right]$$

$$\begin{aligned}
H^{[0]}_{\times} &= -2C\sin 2\Psi, \\
H^{[1/2]}_{\times} &= -\frac{3}{4}SC\sin\Psi + \frac{9}{4}SC\sin 3\Psi, \\
H^{[1]}_{\times} &= \frac{1}{3}C\left[(17-4C^2)-(13-12C^2)\eta\right]\sin 2\Psi - \frac{8}{3}(1-3\eta)S^2C\sin 4\Psi, \\
H^{[3/2]}_{\times} &= \frac{1}{96}SC\left[(63-5C^2)-2(23-5C^2)\eta\right]\sin\Psi \\
&\quad - \frac{9}{64}SC\left[(67-15C^2)-2(19-15C^2)\eta\right]\sin 3\Psi + \frac{625}{192}(1-2\eta)S^3C\sin 5\Psi,
\end{aligned}$$

$$C = \cos\iota$$

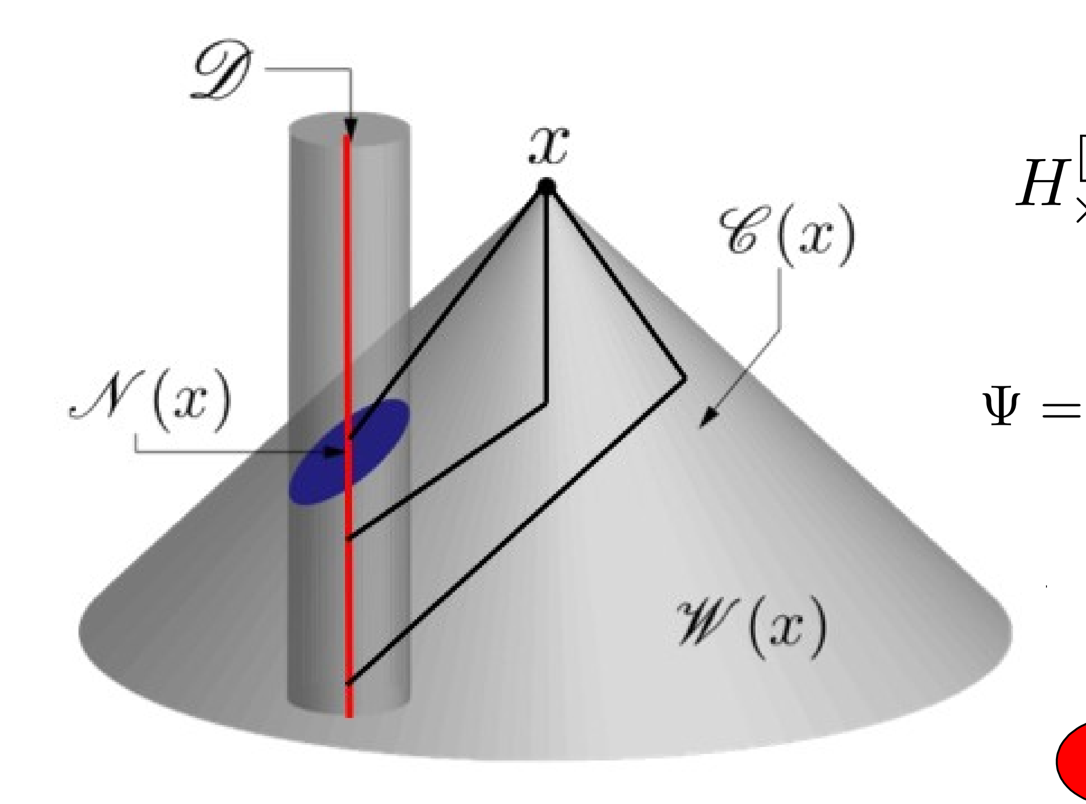
$$S = \sin\iota$$

$$\beta = \left(\frac{Gm\Omega}{c^3}\right)^{1/3} \sim \frac{v}{c}, \quad m = M_1 + M_2\,,\ \eta = \frac{M_1 M_2}{(M_1+M_2)^2}\,,\ \Delta = \frac{M_1 - M_2}{M_1 + M_2}$$

Wave Zone Physics: Gravitational Waves

Beyond the quadrupole formula:

$$h_{+,\times} = \frac{2\eta G m}{c^2 R}\beta^2 \left[(1 + 2\pi\beta^3) H^{[0]}_{+,\times} + \Delta\beta H^{[1/2]}_{+,\times} + \beta^2 H^{[1]}_{+,\times} + \Delta\beta^3 H^{[3/2]}_{+,\times} + O(\beta^4) \right]$$



$$H^{[0]}_{\times} = -2C \sin 2\Psi,$$

$$\Psi = \Omega\left(t - R/c - \frac{2Gm}{c^3} \ln \frac{4\Omega R}{c} \right)$$

Shapiro time delay

Wave Zone Physics: Energy flux

$$
\begin{aligned}
\frac{dE}{dt} &= c \int \partial_0 \tau^{00} d^3x \\
&= -c \oint (-g) t^{0j}_{\rm LL} dS_j \\
&= -\frac{c^3 R^2}{16\pi G} \oint \left[(\partial_t h_+)^2 + (\partial_t h_\times)^2 \right] d\Omega \\
&= -\frac{G}{5c^5} \dddot{I}^{\langle pq \rangle} \dddot{I}^{\langle pq \rangle} + O(c^{-7})
\end{aligned}
$$

- Called the quadrupole formula for energy flux
- Also known as the "Newtonian" order contribution
- Also a flux of angular momentum d**J**/dt and of linear momentum d**P**/dt

For a 2-body system:

$$
\frac{dE}{dt} = \frac{8}{15} \eta^2 \frac{c^3}{G} \left(\frac{Gm}{c^2 r} \right)^4 \left(12v^2 - 11\dot{r}^2 \right),
$$

Energy flux: eccentric orbit

$$\frac{dE}{dt} = \frac{32}{5}\eta^2\frac{c^5}{G}\left(\frac{Gm}{c^2p}\right)^5(1+e\cos\phi)^4\left[1+2e\cos\phi+\frac{1}{12}e^2(1+11\cos^2\phi)\right]$$

$\frac{dE}{dt}$

t/P

Energy flux and binary pulsars

Orbit-averaged flux

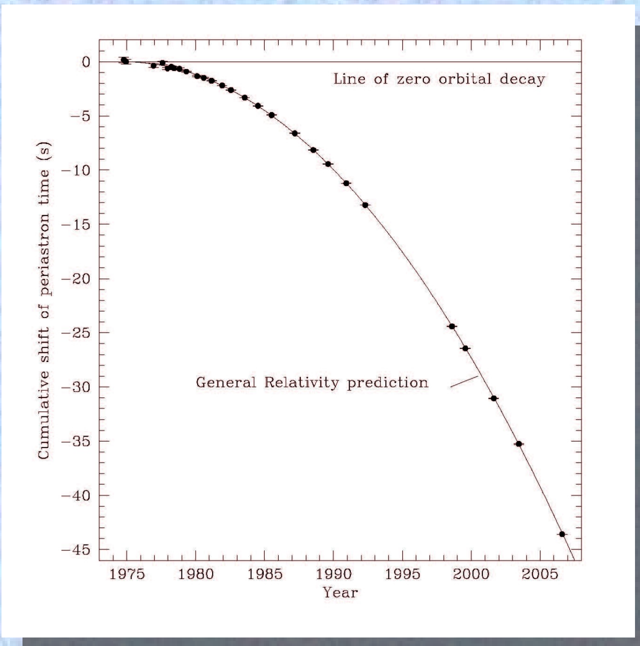
$$\frac{dE}{dt} = \frac{32}{5}\eta^2 \frac{c^5}{G}\left(\frac{Gm}{c^2 a}\right)^5 F(e) \qquad F(e) = \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}$$

Period decrease $E \propto a^{-1} \propto P^{-2/3}$

$$\frac{dP}{dt} = -\frac{192\pi}{5}\left(\frac{G\mathcal{M}}{c^3}\frac{2\pi}{P}\right)^{5/3} F(e)$$

$$\mathcal{M} \equiv \eta^{3/5} m = \text{chirp mass}$$

- ❑ "Newtonian" GW flux
- ❑ 2.5 PN correction to Newtonian equations of motion
- ❑ PN corrections can be calculated, now reaching 4 PN order



PSR 1913+16
Hulse-Taylor binary pulsar

Energy flux and GW interferometers

For a circular orbit, to 3.5 PN order:

$$\nu = \eta = M_1 M_2/(M_1+M_2)^2 \qquad x = \beta^{2/3} = (Gm\Omega/c^3)^{2/3} \sim (v/c)^2$$

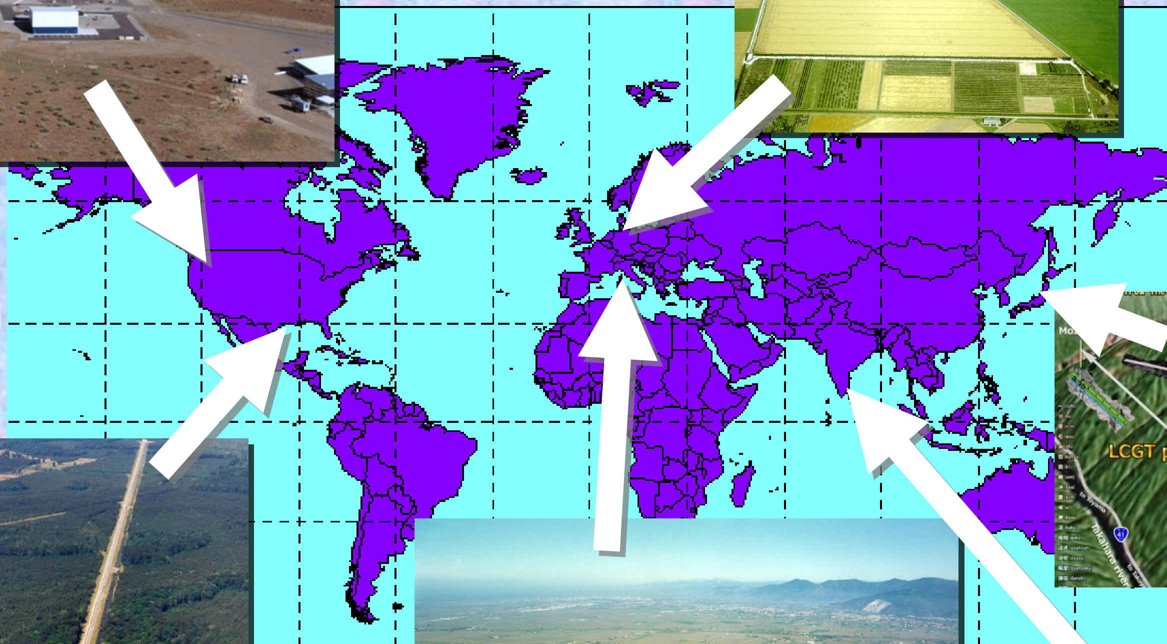
$$\begin{aligned}\frac{dE}{dt} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 &+ \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} \\
&+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
&+ \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln(16x)\right. \\
&\qquad \left. + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\
&+ \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.\end{aligned}$$

From Blanchet, *Living Reviews in Relativity* 17, 2 (2014)

Energy flux and GW interferometers

LIGO Hanford 4&2 km

GEO Hannover 600 m



Kagra Japan
3 km

LIGO Livingston 4 km

Virgo Cascina 3 km

LIGO South
Indigo

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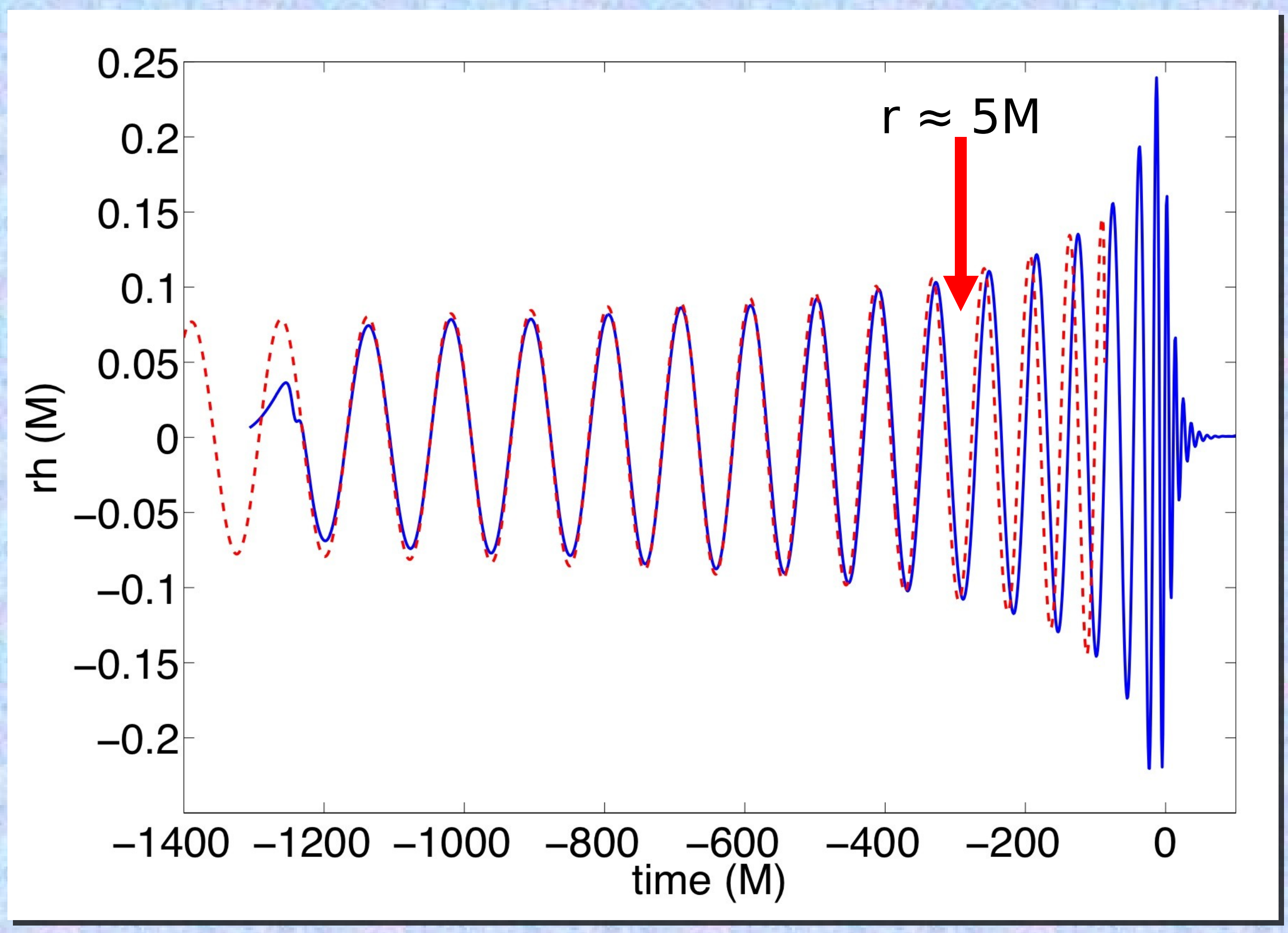
Energy flux and GW interferometers



Baker *et al.* gr-qc/0612024

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Wave Zone Physics: Radiation reaction

Loss of energy at order c^{-5} implies that the dynamics of a system cannot be conservative at 2.5 PN order

There must be a **radiation reaction force F** that dissipates energy according to

$$\sum_A \boldsymbol{F}_A \cdot \boldsymbol{v}_A = \frac{dE}{dt}$$

To find this force, we return to the near-zone and iterate the relaxed Einstein equations **3 times** to find the metric to 2.5 PN order

That metric is inserted into the equations of motion $\nabla_\beta T^{\alpha\beta} = 0$

There are Newtonian, 1 PN, 2 PN, 2.5 PN, …. terms (no 1.5 PN!)
Happily, to find the leading 2.5 PN contributions, it is not necessary to calculate the 2 PN terms explicitly (though that has been done)

Wave Zone Physics: Radiation reaction

Newtonian plus corrections up to 2.5 PN order within τ^{00}

expand h^{00} in the near zone:

No 0.5 PN term: conservation of M

1 PN correction $d^{2X/dt2}$

$$h^{00}_{\mathcal{N}} = \frac{4G}{c^4}\Bigg\{\int_{\mathcal{M}} \frac{\tau^{00}}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x'}|\, d^3x'$$

$$\cancel{-\frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x'}|^2\, d^3x'} + \frac{1}{24c^4}\frac{\partial^4}{\partial t^4}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x'}|^3\, d^3x'$$

$$-\frac{1}{120c^5}\frac{\partial^5}{\partial t^5}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x'}|^4\, d^3x'\Big] + O(c^{-6})\Bigg\}$$

2 PN term

Pure function of time – a coordinate effect

2.5 PN term

Wave Zone Physics: Radiation reaction

Pulling all the contributions together, we find the equations of hydrodynamics to 2.5 PN order

$$\rho^* \frac{d\boldsymbol{v}}{dt} = \rho^* \boldsymbol{\nabla} U - \boldsymbol{\nabla} p + O(c^{-2}) + O(c^{-4}) + \boldsymbol{f}$$

Where f is a radiation reaction force density. For body A

$$\boldsymbol{F}_A = \int_A \rho^* \boldsymbol{f} d^3x$$

For a 2-body system, this leads to a radiation-reaction contribution

$$\boldsymbol{a}[\mathrm{rr}] = \frac{8}{5}\eta \frac{(GM)^2}{c^5 r^3}\left[\left(3v^2 + \frac{17}{3}\frac{GM}{r}\right)\dot{r}\,\boldsymbol{n} - \left(v^2 + 3\frac{GM}{r}\right)\boldsymbol{v}\right]$$

This is harmonic gauge (also called Damour-Deruelle gauge)

Wave Zone Physics: Radiation reaction

Alternative gauge: the Burke-Thorne gauge. All RR effects embodied in a modification of the Newtonian potential

$$U \to U - \frac{G}{5c^5}\frac{d^5 I^{\langle jk \rangle}}{dt^5} x^j x^k$$

For a two body system

$$\boldsymbol{a}[\mathrm{rr}] = \frac{8}{5}\eta\frac{(GM)^2}{c^5 r^3}\left[\left(18v^2 + \frac{2}{3}\frac{GM}{r} - 25\dot{r}^2\right)\dot{r}\,\boldsymbol{n} - \left(6v^2 - 2\frac{GM}{r} - 15\dot{r}^2\right)\boldsymbol{v}\right]$$

In any gauge, orbital damping precisely matches wave-zone fluxes:

$$\frac{dE}{dt} = \frac{8}{15}\eta^2\frac{c^3}{G}\left(\frac{Gm}{c^2 r}\right)^4\left(12v^2 - 11\dot{r}^2\right),$$

$$\frac{dJ^j}{dt} = \frac{8}{5}\eta^2\frac{c}{G}\left(\frac{Gm}{c^2 r}\right)^3 h^j\left(2v^2 - 3\dot{r}^2 + 2\frac{Gm}{r}\right),$$

$$\frac{dP^j}{dt} = -\frac{8}{105}\Delta\eta^2\frac{c}{G}\left(\frac{Gm}{c^2 r}\right)^4\left[v^j\left(50v^2 - 38\dot{r}^2 + 8\frac{Gm}{r}\right) - \dot{r}n^j\left(55v^2 - 45\dot{r}^2 + 12\frac{Gm}{r}\right)\right]$$

Wave Zone Physics: Radiation reaction

Inserting $\mathbf{a}_{RR}$ into the Lagrange planetary equation as a disturbing force and integrating over an orbit

$$\frac{dp}{dt} = -\frac{64}{5}\eta c\left(\frac{GM}{c^2 p}\right)^3 (1-e^2)^{3/2}\left(1+\frac{7}{8}e^2\right),$$

$p=a(1-e^2)$

$$\frac{de}{dt} = -\frac{304}{15}\eta c\frac{e}{p}\left(\frac{GM}{c^2 p}\right)^3 (1-e^2)^{3/2}\left(1+\frac{121}{304}e^2\right)$$

Radiation reaction causes 2-body orbits to inspiral and circularize

- The Hulse-Taylor binary pulsar will circularize and merge within 300 Myr; the double pulsar within 85 Myr
- This is short compared to the age of galaxies (5 – 10 Gyr)
- There must be NS-NS binaries merging today (possibly even NS-BH and BH-BH binaries)
- The inspiral of compact binaries is a leading potential source of GW for interferometers

Outline of the Lectures*

Part 1: Newtonian Gravity

- Foundations
- Equations of hydrodynamics
- Spherical and nearly spherical bodies
- Motion of extended fluid bodies

Part 2: Newtonian Celestial Mechanics

- Two-body Kepler problem
- Perturbed Kepler problem

*Based on *Gravity: Newtonian, post-Newtonian, General Relativistic,* by Eric Poisson and Clifford Will (Cambridge U Press, 2014)

Outline of the Lectures*

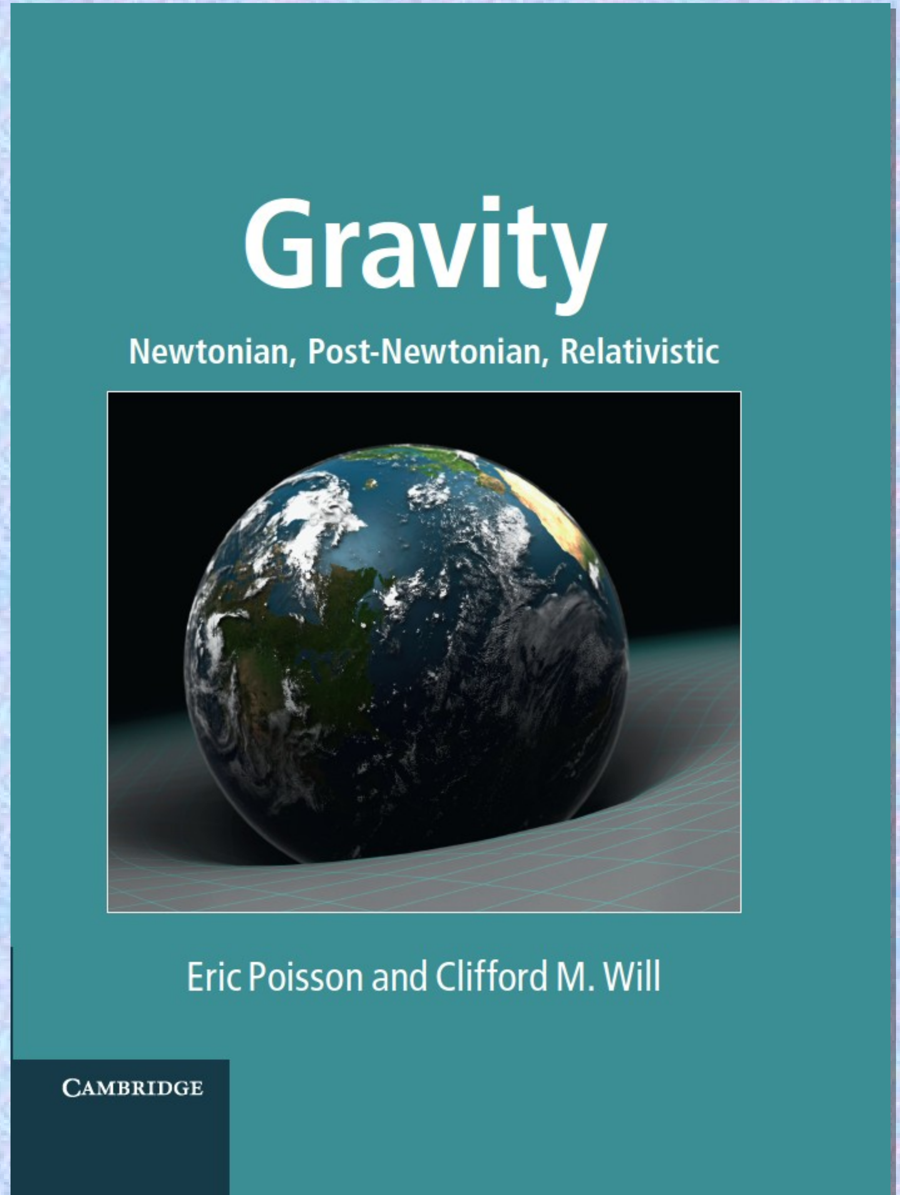
Part 3: General Relativity

- Einstein equivalence principle
- GR field equations

Part 4: Post-Newtonian & post-Minkowskian theor

- Formulation
- Near-zone physics
- Wave-zone physics
- Radiation reaction

*Based on *Gravity: Newtonian, post-Newtonian, General Relativistic,* by Eric Poisson and Clifford Will (Cambridge U Press, 2014)



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